GR Sessions 7: Black Holes

Wednesdays November 21, 2012

1. **Conformal diagrams.** The solution to Einstein's equations with a positive cosmological constant can be written as

$$\begin{aligned} ds^2 &= \ell^2 \left(-d\tau^2 + \cosh^2\tau \left(d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)\right) \\ &= \ell^2 \left(-d\tau^2 + \cosh^2\tau \, d\Omega_3^2\right) \ , \end{aligned}$$

where ψ and θ range from $[0, \pi]$ and ϕ ranges from $[0, 2\pi]$. The quantity in parentheses is the metric on the 3-sphere. Using the coordinate transformation $\cosh\tau = \sec T$, draw the Penrose (conformal) diagram of de Sitter space. You can find information about Penrose diagrams in Appendix H of Carroll.

2. **Kerr black holes** An observer orbits a Kerr black hole of Mass M and angular momentum (per unit mass) a in the equatorial plane.

   (a) Consider a constant r orbit and define $\Omega = \frac{d\phi}{dt}$ to be her angular velocity as measured by a very distant and stationary observer. Show that the observer's four velocity is given by

   $$v^\mu = v^0(1, 0, 0, \Omega) \ ,$$

   where

   $$v^0 = \left(1 - \frac{2GM}{r} + \frac{4GMa}{r}\Omega - \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\Omega^2\right)^{-1/2} .$$

   (b) Consider the polynomial

   $$Y \equiv -1 + \frac{2GM}{r} - \frac{4GMa}{r}\Omega + \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\Omega^2 \ .$$

   Using part 2a show that Y is always negative.

   (c) Using this result show that Ω is nonzero in the ergosphere. Also show that the observer can not stay fixed at constant radius once she crosses the outer horizon r_+.

   (d) Show that Kepler's law $\Omega^2 = \frac{GM}{r^3}$ holds for circular orbits around a Schwarzschild black hole.

   (e) Derive an analogous result for equatorial orbits around a Kerr black hole. *Hint: You can save a lot of time by first showing the geodesic equation reduces to*

   $$\Gamma_{\mu\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \ ,$$

   where $\Gamma_{\mu\nu\rho} = (\partial_\nu g_{\mu\rho} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho})$.