GR Sessions 9: Gravitational Perturbations

Wednesdays December 5, 2012

1. **Linearized Einstein's equations (Carroll 7.1).** Show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu h^{\mu\nu})(\partial_\nu h) - (\partial_\mu h^{\rho\sigma})(\partial_\rho h^\mu_\sigma) + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu h)(\partial_\nu h)\right] ,$$

gives rise to the linearized version of Einstein's equation.

2. **Diffeomorphisms.** Show that if we decompose the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then the components of the Riemann tensor are

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu\, h_{\mu\sigma} + \partial_\sigma\partial_\mu\, h_{\nu\rho} - \partial_\sigma\partial_\nu\, h_{\mu\rho} - \partial_\rho\partial_\mu\, h_{\nu\sigma}\right) ,$$

to linear order in $h_{\mu\nu}$. Show explicitly that this linearized Riemann tensor is invariant under the gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\, \xi_\nu + \partial_\nu\, \xi_\mu \ .$$

3. **Plane waves (Carroll 7.5)** Consider the metric

$$ds^2 = -(du\, dv + dv\, du) + a^2(u)dx^2 + b^2(u)dy^2 ,$$

where a and b are arbitrary functions of u. For appropriate functions a and b this represents an *exact* gravitational plane wave.

   (a) Calculate the Chistoffel symbols and Riemann tensor for this metric.

   (b) Use Einstein's equations in vacuum to derive equations obeyed by $a(u)$ and $b(u)$.

   (c) Show that an exact solution can be found, in which both a and b are determined in terms of an *arbitrary* function $f(u)$.