analytical mechanics
12 June 2017 AM
Name: ....

(1) Consider a simple pendulum where the angle $\theta(t)$ varies with maximum $\pm\theta_0$. What is the probability density $\rho(\theta)$?

(2) Consider a general scaling transformation

$$Q = \Lambda q, \quad P = \Lambda' p$$

where Λ and Λ' are real symmetric positive matrices. Show that that transformation is canonical if and only if $\Lambda' = \Lambda^{-1}$.

(3) Use the method of Hamilton-Jacobi to treat a simple harmonic oscillator in two dimensions.
What is the Hamilton-Jacobi equation here?
Solve it.
Use it to give the positions as functions of time and of the initial conditions.

(4) Prove the Young inequality, that for all $\alpha, \beta \geq 1, x, p > 0$,

$$px \leq \frac{x^\alpha}{\alpha} + \frac{p^\beta}{\beta}, \qquad \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

(5) A bead of mass m slides without friction on a circular loop of radius a. The loop lies in a vertical plane and rotates about a vertical diameter with constant angular velocity ω. Gravity acts.
a) For angular velocity $\omega > \omega_c$ greater than some critical value, the bead can undergo small oscillations about some stable equilibrium point θ_0. Find ω_c and $\theta_0(\omega)$.
b) Obtain the equations of motion for the small oscillations about θ_0 as a function of ω and find the period of these oscillations.

(6) Show that the logistic map
$x \mapsto r\,x(1 - x)$ on $[0, 1]$
has a two-cycle for all $r > 3$ and discuss its stability.

(7) Consider the motion of a perfectly elastic rigid ball of fixed mass m between two perfectly elastic walls whose separation ℓ slowly varies. Prove that the product $v\,\ell$ of the product of the speed of the ball and the separation distance is an adiabatic invariant.