analytical mechanics
6 June 2016 AM
Name: ....

1. A simple pendulum in the earth's gravitational field consists of a mass $M = 1$ kg suspended by a thin, massless string of 1m. Compute the tension in the string as a function of the angle.

2. Consider the mechanical motion of a particle in one dimensional space under a potential $V(x) = -kx^2/2 + ax^4/4$, function of small parameter $a > 0$.
a) Draw the possible orbits in phase space (x, p) [phase portrait].
b) Give the period of the motion in linear order in a.

3. Write down the Lagrangian for the following system: a cart of mass m can roll without friction on a rail along the x-axis. A pendulum, consisting of a stick of length ℓ and a point mass m, is mounted rigidly on the cart and can move freely within the $x - z$ vertical plane.

4. For proving the Euler-Lagrange equation from the variation of the action, we need to know that, if for all real-valued functions u which are sufficiently smooth

$$\int_a^b \mathrm{d}x\, u(x)\, w(x) = 0$$

with w also smooth, then in fact, $w = 0$. Show that.

5. Show that the Hamiltonian flow is itself a canonical transformation.

6. Give the Liouville equation for the smooth dynamical system
$\dot{x}(t) = f(x(t)), x(t) \in \mathbb{R}^n$.

7. Use the method of Hamilton-Jacobi to treat a simple harmonic oscillator in two dimensions.
What is the Hamilton-Jacobi equation here?
Solve it.
Use it to give the positions as functions of time and of the initial conditions.

8. Show there is no periodic motion for one-dimensional dynamical systems $\dot{x}(t) = f(x(t))$, $x(t) \in \mathbb{R}$, and no limit cycle is possible.
Show that there cannot be chaos for two-dimensional dynamical systems $\dot{x}(t) = f(x(t))$, $x(t) \in \mathbb{R}^2$

9. Show that the periodic points of the Bernoulli shift
$x \mapsto 2x$ mod 1 on $[0, 1]$
are dense in $[0, 1]$.

10. Show that the logistic map
$x \mapsto r\,x(1-x)$ on $[0, 1]$
has a two-cycle for all $r > 3$ and discuss its stability.