# Commutative algebra  Exam 2012-2013

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January 15, 2013

### Instructions

You have four and a half hours for this exam. It consists of two separate parts: an oral part with theoretical questions and a written part with exercises. After (at most) one hour of preparation, you should be ready for the oral part of the exam. This exam will be graded by means of a score $E$ out of 20 marks. Together with the homework score $H$ (out of 20 marks), this leads to a final score $F$ given by the formula $F = \max\left\{E, \frac{1}{4}(3E+H)\right\}$.

In the written part, just as for the homework assignments, we sometimes put the 'difficulty indicators' $\circ$ and $\star$ .

**Important:** During this exam, 'ring' means 'commutative ring with a unit element'.

**Good luck!**

## Oral part

**Question 1 -** Write down an explicit proof of Theorem 7.33.

**Question 2 -** Explain and/or answer questions about the proof of Lemma 9.32.

**Question 3 -** Surprise...

## Written part

**Question 1 - "Prime factorization in Dedekind rings"**

*This question builds on the material developed in homework 2 and homework 4.*

Let $R$ be a Dedekind ring. The following paragraph provides a proof for the fact that every non-trivial ideal of $R$ is a product of finitely many prime ideals. However, many details are left out. Provide full details where you think this is necessary.

> Let $\mathfrak{a}$ be a non-trivial ideal of $R$. Because $R$ is Noetherian, it is sufficient to prove the statement if we assume that every ideal strictly bigger than $\mathfrak{a}$ is a product of finitely many prime ideals. Let $\mathfrak{m} \supseteq \mathfrak{a}$ be a maximal ideal. Then $\mathfrak{a} \subseteq \mathfrak{a} \cdot (R : \mathfrak{m}) \subseteq R$. If $\mathfrak{a} \cdot (R : \mathfrak{m}) = \mathfrak{a}$, then $\mathfrak{a} = \mathfrak{a} \cdot \mathfrak{m}$, but this is impossible because of Nakayama's lemma. Hence $\mathfrak{a} \cdot (R : \mathfrak{m}) \neq \mathfrak{a}$. Since $\mathfrak{a} \cdot (R : \mathfrak{m})$ is a product of finitely many prime ideals by assumption, the same is true for $\mathfrak{a}$. $\square$

**Question 2 - "Support of a module"**

Let $R$ be a ring, $M$ a finitely generated $R$-module and $\mathfrak{p} \subseteq R$ a prime ideal. Show that $M_\mathfrak{p} \neq 0$ if and only if $\mathfrak{p} \supseteq \operatorname{Ann} M$.

**Question 3 - "A concrete basis"**

Let $S$ be a domain which contains a principal ideal domain $R$ (as a subring) such that $S$ is a finitely generated $R$-module.

(a) Prove that $S$ is free as an $R$-module.

(b) Consider the special case $S = \mathbb{C}[X,Y]/(X^2 - Y^3) = \mathbb{C}[x,y]$ (where $x = \overline{X}$, $y = \overline{Y}$) and $R = \mathbb{C}[t]$, where $t = x^3y^4$. Give an explicit basis for $S$ as an $R$-module. (You don't need to prove that this is a basis.)

**Question 4 - "Representability of the nilradical"**

Let $\mathsf{Ring}$ be the category of commutative rings.

(a) Show that for each $n \geq 1$, there is a covariant functor $\mathsf{Nil}_n : \mathsf{Ring} \to \mathsf{Set}$ which maps a ring $R$ to $\{x \in R : x^n = 0\}$. Similarly, show that there is a covariant functor $\mathsf{Nil} : \mathsf{Ring} \to \mathsf{Set}$ which maps a ring $R$ to its nilradical. ($\circ$)

(b) Show that for each $n \geq 1$, the functor $\mathsf{Nil}_n$ is represented by the ring $\mathbb{Z}[X]/(X^n)$ and the natural isomorphism

$$\tau : h_{\mathbb{Z}[X]/(X^n)} \to \mathsf{Nil}_n$$

given (for each ring $R$) by

$$\tau_R : h_{\mathbb{Z}[X]/(X^n)}(R) \to \mathsf{Nil}_n(R) : f \mapsto f(\overline{X}).$$

(c) Show that $\mathsf{Nil}$ is *not* representable. ($\star$)

**Question 5 - "Finitely generated modules over local rings"**

Let $A$ be a ring and let $M$ be a finitely generated $A$-module. If $S$ is a set of generators of $M$ which is minimal in the sense that any proper subset of $S$ does not generate $M$ as an $A$-module, then $S$ is said to be a *minimal generating set* for $M$ over $A$.

(a) Show (using an example) that minimal generating sets for $M$ over $A$ need not have the same number of elements. ($\circ$)

Assume from now on that $A$ is a local ring. Let $\mathfrak{m}$ be its maximal ideal and $k = A/\mathfrak{m}$ its residue field.

(b)
  (1) Explain (very briefly) why $M/\mathfrak{m}M$ is a finite dimensional vector space over $k$. ($\circ$)
  (2) Let $\{\overline{u_1}, \cdots, \overline{u_n}\}$ be a basis for $M/\mathfrak{m}M$ over $k$, and choose for $1 \leq i \leq n$ an element $u_i \in M$ which maps to $\overline{u_i}$ in $M/\mathfrak{m}M$. Show, using Nakayama's lemma, that $u_1, \cdots, u_n$ is a minimal generating set for $M$ over $A$. ($\circ$)
  (3) Conversely, show that every minimal generating set for $M$ over $A$ can be obtained via the construction described in (2). Conclude from this that every minimal generating set for $M$ over $A$ has the same number of elements. ($\circ$)

We will now prove that if $M$ is projective, then $M$ is free.

(c) Let $\{u_1, \cdots, u_n\}$ be a minimal generating set for the projective module $M$ over $A$. Define $\varphi : F = A^n \to M$ via $e_i \mapsto u_i$. (Here $e_1, \ldots, e_n$ are the standard basis of $A^n$.)
  (1) Let $K = \ker(\varphi)$. Show that $K \subseteq \mathfrak{m}F$.
  (2) Show that there exists a homomorphism of $A$-modules $\psi : M \to F$ such that $F = \psi(M) \oplus K$.
  (3) Deduce that $K = \mathfrak{m}K$.
  (4) Conclude, again using Nakayama's lemma, that $M \cong F$. Hence $M$ is indeed a free $A$-module.