General Relativity - Exercise session

Friday October 18, 2013

1. Using the Ricci identity, *i.e.* the relation on the non-commutativity of second covariant derivatives used as definition of the Riemann tensor (see eq 3.114), prove that if $A^{\mu\nu}$ is antisymmetric and the connection is torsion free, then:
$$\nabla_{[\mu}\nabla_{\nu]}A^{\mu\nu} = 0.$$

2. Let u^μ be a vector field tangent to a geodesic with affine parameter λ and ξ^μ a Killing vector field. Show that $u^\mu \xi_\mu$ is a constant along the geodesic.

3. Consider the metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ on the two sphere.
   (a) Which isometry is manifest in this coordinates and what is the associated Killing vector?
   (b) Prove that $\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi$ is another Killing vector.
   (c) Remember that the commutator of two Killing vectors is a Killing vector. Use this property to find a third linearly independent Killing vector.

4. Compute Christoffel symbols, Riemann tensor and Ricci tensor for the following line element
$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2$$
where the functions α and β depend on the coordinate r only.