Exam Functional Analysis

September 5, 2013

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for **4 hours.** You are allowed to eat or drink.
- After **2 hours,** you hand in your solutions for questions 1 and 2. During the third and fourth hour, you work on questions 3 and 4, and you will have your oral exam about questions 1 and 2. **After 4 hours, the exam ends.**
- The exam is **open book.** This means that you may use
  - the lecture notes,
  - your own notes,
  - the two reference books.

  You are **not allowed to use**
  - any electronic equipment,
  - other books than the two reference books.
- The four questions have the same weight.

Write your name on every sheet that you hand in !

Good luck ! Stefaan Vaes

1. Let H be a Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Denote by P_n the orthogonal projection of H onto $\text{span}\{e_0, \ldots, e_n\}$. Let $K \subset H$ be a compact subset. Prove that
$$\lim_{n\to\infty} P_n\xi = \xi \quad \text{uniformly in } \xi \in K.$$
Is the convergence also uniform over all ξ in the unit ball of H ?

2. Make Exercise 1 in the beginning of Section 8.3. So you have to prove the following statement.
Let X be a Banach space. Denote by $i : X \to X^{**}$ the isometry defined by $i(x) : X^* \to \mathbb{C} : \omega \mapsto \omega(x)$ for all $x \in X$. Prove that the image $i((X)_1)$ of the unit ball of X is weak dense in the unit ball $(X^{**})_1$ of X^{**}.*
As explained in Section 8.3, you have to proceed as follows. Denote by K the weak* closure of $i((X)_1)$ inside X^{**}. Suppose that $\theta \in X^{**} \setminus K$. You have to prove that $\|\theta\| > 1$. Do this by applying the Hahn-Banach separation theorem to K and $\{\theta\}$.

3. Let $(X, \mathcal{P})$ be a seminormed space and equip X with the seminorm topology. Let $K \subset X$ be a compact subset and let $\mathcal{U} \subset X$ be a neighborhood of 0. For every $\lambda > 0$, define
$$\lambda\mathcal{U} = \{\lambda x \mid x \in \mathcal{U}\} .$$
Prove that there exists a $\lambda_0 > 0$ such that $K \subset \lambda\mathcal{U}$ for all $\lambda > \lambda_0$.
Hint: we used a variant of this in the proof of the Markov-Kakutani fixed point theorem.

4. Consider the Banach space $\ell^\infty(\mathbb{N})$ with its supremum norm. Denote by $1 \in \ell^\infty(\mathbb{N})$ the function that is equal to 1 everywhere. Define the subset $K \subset \ell^\infty(\mathbb{N})^*$ given by
$$K = \left\{\omega \in \ell^\infty(\mathbb{N})^* \;\middle|\; \omega(1) = 1, \|\omega\| \leq 1, \omega(FG) = \omega(F)\,\omega(G) \;\text{ for all }\; F, G \in \ell^\infty(\mathbb{N})\right\} .$$
   a) Prove that K is compact in the weak* topology on $\ell^\infty(\mathbb{N})^*$.
   b) For every subset $A \subset \mathbb{N}$, define $\chi_A \in \ell^\infty(\mathbb{N})$ given by
   $$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$
   Prove that for every $A \subset \mathbb{N}$ and every $\omega \in K$, we have that $\omega(\chi_A) \in \{0, 1\}$.
   c) Fix $n \in \mathbb{N}$ and define $\delta_n \in K$ given by $\delta_n(F) = F(n)$ for all $F \in \ell^\infty(\mathbb{N})$. Prove that the singleton $\{\delta_n\}$ is an open subset of K in the weak* topology.
   Hint: use the function $\chi_{\{n\}}$.