Exam Functional Analysis

January 16, 2014

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for **4 hours.** You are allowed to eat or drink.
- After **1 hour,** you hand in your solutions for question 1. During the remaining time, you work on questions 2, 3 and 4, and you will have your oral exam about question 1. **After 4 hours, the exam ends.**
- The exam is **open book.** This means that you may use
  - the lecture notes,
  - your own notes,
  - the two reference books.

  You are **not allowed to use**
  - any electronic equipment,
  - other books than the two reference books.
- This part of the exam counts for 12 of the 20 points. Every of the four questions has the same weight. The other 8 of the 20 points are attributed on the take home exam.

Write your name on every sheet that you hand in !

Good luck ! Stefaan Vaes

1. Consider for $1 \leq p < +\infty$ the Banach space $X = \ell^p(\mathbb{N})$ with the norm $\| \cdot \|_p$. For which values of p, the unit ball $\{\xi \in X \mid \|\xi\|_p \leq 1\}$ is weakly compact ? Give a proof for your answer.

2. a) Prove that the maps T_g that are used in the proof of Theorem 9.4 are indeed weak* continuous.

   b) Prove Proposition 9.5.

3. Let X be a topological vector space over $\mathbb{R}$. Let $\omega : X \to \mathbb{R}$ be a linear map. Let $\alpha \in \mathbb{R}$. Prove that the following three statements are equivalent.

   a) The set $\{x \in X \mid \omega(x) < \alpha\}$ is open.

   b) The set $\{x \in X \mid \omega(x) \leq \alpha\}$ is closed.

   c) The map ω is continuous.

4. Let G be a countable, amenable group. Let G act on a countable set I. We denote the action of $g \in G$ on $x \in X$ as $g \cdot x$. Whenever $\xi : I \to \mathbb{C}$ is a function and $g \in G$, we denote by $\xi \cdot g : I \to \mathbb{C}$ the translated function given by $(\xi \cdot g)(x) = \xi(g \cdot x)$.

   a) Prove that there exists a G-invariant mean on I, i.e. a finitely additive probability measure on I satisfying $m(g \cdot A) = m(A)$ for all $g \in G$ and $A \subset I$.

   b) Prove that there exists a sequence of finitely supported functions $\xi_n : I \to [0, +\infty)$ satisfying

$$\sum_{x \in I} \xi_n(x) = 1 \text{ for all } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \to \infty} \|\xi_n \cdot g - \xi_n\|_1 = 0 \text{ for all } g \in G.$$