Exam Functional Analysis

September 3, 2014

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for **4 hours.** You are allowed to eat or drink.
- After **2 hours,** you hand in your solution for question 1 and I will start the oral exam about question 1. **After 4 hours, the exam ends** and you hand in your written solutions for questions 2, 3 and 4.
- The exam is **open book.** This means that you may use
  - the lecture notes,
  - your own notes,
  - the two reference books.

  You are **not allowed to use**
  - any electronic equipment,
  - other books than the two reference books.
- Every of the four questions has the same weight in your total score.

Write your name on every sheet that you hand in !

Good luck ! Stefaan Vaes

1. Let H be a Hilbert space. For every $T \in B(H)$ and $\lambda \in \mathbb{C}$, we define the λ-eigenspace $H(T, \lambda)$ of T as the set of all eigenvectors of T with eigenvalue λ.

   a) Let $S, T \in B(H)$ be bounded operators that commute : $ST = TS$. Prove that $S(H(T, \lambda)) \subset H(T, \lambda)$ for all $\lambda \in \mathbb{C}$.

   b) Let S and T be compact self-adjoint operators on H. Prove that $ST = TS$ if and only if there exists an orthonormal basis of H consisting of vectors that are eigenvectors for both S and T.

2. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $x : \mathbb{N} \to \mathbb{C}$ be a sequence with the property that $xy \in \ell^1(\mathbb{N})$ for all $y \in \ell^q(\mathbb{N})$. Prove that $x \in \ell^p(\mathbb{N})$.

3. Let X be a seminormed space with its seminorm topology. Suppose that A and B are nonempty disjoint convex subsets of X such that A is compact and B is closed. Prove that there exists an open convex subset $A_1 \subset X$ such that $A \subset A_1$ and $A_1 \cap B = \emptyset$.
   This means that you have to write all the details for the first paragraph of the proof of Corollary 8.5, including how exactly (8.1) is used.

4. Let G be a countable, amenable group. Let G act on a countable set I. We denote the action of $g \in G$ on $x \in X$ as $g \cdot x$. Whenever $\xi : I \to \mathbb{C}$ is a function and $g \in G$, we denote by $\xi \cdot g : I \to \mathbb{C}$ the translated function given by $(\xi \cdot g)(x) = \xi(g \cdot x)$.

   a) Prove that there exists a G-invariant mean on I, i.e. a finitely additive probability measure on I satisfying $m(g \cdot A) = m(A)$ for all $g \in G$ and $A \subset I$.

   b) Prove that there exists a sequence of finitely supported functions $\xi_n : I \to [0, +\infty)$ satisfying

$$\sum_{x \in I} \xi_n(x) = 1 \text{ for all } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \to \infty} \|\xi_n \cdot g - \xi_n\|_1 = 0 \text{ for all } g \in G.$$