Take home exam Functional Analysis

- The deadline to hand in your solutions for the take home exam is **Friday, December 14 at 12h00.** You can either send a pdf file to stefaan.vaes@wis.kuleuven.be or you can put your solutions in the mailbox of Stefaan Vaes on the ground floor of the mathematics department.
- The take home exam counts for 8 of the 20 points. The exam in January 2013 counts for 12 of the 20 points.
- You may choose whether you write or type your solutions. This does not influence the evaluation. It is up to you whether you write or type your solutions in English or in Dutch.
- You may discuss with your fellow students about the take home exam. But everybody has to write down his own solutions, meaning that literally the same solutions will not be tolerated. Of course, the same, or similar, methods may be used. During the oral exam in January 2013, I might also ask you questions about your solutions of the take home exam.

1. Let $\lambda : \mathbb{N} \to \mathbb{C}$ be a bounded function. Define the operator

$$W : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) : (Wx)(n) = \lambda(n)\, x(n+1) \;\text{ for all }\; x \in \ell^2(\mathbb{N}) \;\text{ and }\; n \in \mathbb{N} .$$

Under which condition on λ, the operator W is compact? Prove your answer.

2. Let X be a normed space. Whenever $\mathcal{F} \subset X$ is a subset, denote

$$\mathcal{F}^\perp := \{\varphi \in X^* \mid \varphi(x) = 0 \;\text{ for all }\; x \in \mathcal{F}\} .$$

Whenever $\mathcal{G} \subset X^*$ is a subset, denote

$$^\perp\mathcal{G} := \{x \in X \mid \varphi(x) = 0 \;\text{ for all }\; \varphi \in \mathcal{G}\} .$$

Prove that for every subset $\mathcal{F} \subset X$, we have that $^\perp(\mathcal{F}^\perp)$ equals the closure of the linear span of $\mathcal{F}$.

3. Let $x_n : \mathbb{N} \to \mathbb{C}$ be a sequence of functions with $x_n \in \ell^1(\mathbb{N})$ for every $n \in \mathbb{N}$. Prove that the following two statements are equivalent.

   - $\sup_n \|x_n\|_1 < \infty$ and for every $j \in \mathbb{N}$, we have $\lim_n x_n(j) = 0$.
   - For every $y \in c_0(\mathbb{N})$, we have that

$$\lim_{n\to\infty} \Big(\sum_{j=0}^{\infty} x_n(j)\, y(j) \Big) = 0 .$$

4. Let $x : \mathbb{N} \to \mathbb{C}$ be a function. Prove that the following two statements are equivalent.

   - $x \in \ell^2(\mathbb{N})$.
   - For every $y \in \ell^2(\mathbb{N})$, we have that $xy \in \ell^1(\mathbb{N})$.

Hint: use the closed graph theorem.