Exam Functional Analysis

KU Leuven, January 28, 2015

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for four hours. You are allowed to eat or drink.
- After one hour, you hand in your solution for Exercise 1. During the rest of the time, you work on the rest of the exercises and you will have your oral exam about Exercise 1.
- The exam is open book. This means that you may use the lecture notes and your own notes. You are not allowed to use anything else.
- This part of the exam counts for 12 of the 20 points. Every of the four exercises has the same weight.
- Write your name on every sheet that you hand in!

Good luck!

Exercise 1.

Let X be a Banach space. We consider X with the weak topology (as defined in Example 7.2 of the lecture notes) and its dual space X^* with the weak* topology (as defined in Example 7.3 of the lecture notes).

(i) Show that every weak* convergent sequence in X^* is bounded.

(ii) Show that every weakly convergent sequence in X is bounded.

Exercise 2.

Consider the space $C([0,1],\mathbb{R})$ of continuous functions from the interval $[0,1]$ to $\mathbb{R}$.

Let $X = \{f \in C([0,1],\mathbb{R}) \mid \text{the derivative } f' \text{ exists and is continuous on } [0,1]\}$, let $Y = C([0,1],\mathbb{R})$, and equip both X and Y with the norm $\|.\|_\infty$. Consider the operator $D\colon X \to Y$ given by $Df = f'$. Clearly, D is a linear operator. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in X such that $(f_n, f_n') \to (f, g)$ in $X \oplus_1 Y$. Then $f_n' \to g$ uniformly on $[0,1]$ as $n \to \infty$.

(i) Show that $f' = g$, and deduce that $\operatorname{graph}(D)$ is closed.

(ii) Show that D is not a bounded operator.
Hint: consider the functions f_n on $[0,1]$ given by $f_n(t) = t^n$.

(iii) Why do (i) and (ii) not contradict the Closed Graph Theorem?

Exercise 3.

Let H be a Hilbert space, and let $B(H)$ be the space of bounded operators on H. In this exercise, we consider the strong topology and the weak topology on $B(H)$, as introduced in Example 7.4 of the lecture notes. Recall that the strong topology is defined by the family $\mathcal{P}_{\text{strong}}$ of seminorms given by

$$\mathcal{P}_{\text{strong}} = \{T \mapsto \|Tx\| \mid x \in H\},$$

and that the weak topology is defined by the family $\mathcal{P}_{\text{weak}}$ of seminorms given by

$$\mathcal{P}_{\text{weak}} = \{T \mapsto |\langle Tx, y\rangle| \mid x, y \in H\}.$$

(Recall that this topology does not coincide with the weak topology on $B(H)$ when viewing $B(H)$ as a Banach space.)

(i) Show that the strong topology is stronger than the weak topology.

(ii) By considering the operator $U_n\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by $(U_n x)(k) = x(n+k)$ for all $x \in \ell^2(\mathbb{Z})$ and $k \in \mathbb{Z}$, show that the strong topology is actually **strictly** stronger than the weak topology.

This exercise is part of Exercise 4 of Chapter 7.

Exercise 4.

This exercise provides a method of showing that certain Banach spaces are not isometrically isomorphic to the dual of another Banach space.

(i) Let X be a Banach space. Show that the closed unit ball of the dual space X^* of X has extreme points.

(ii) Consider the Banach space $c_0(\mathbb{N}) = \{x \colon \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} x(n) = 0\}$ (equipped with the norm $\|.\|_\infty$). Show that its closed unit ball does not have any extreme points, and conclude that $c_0(\mathbb{N})$ cannot be the dual of any Banach space.

(iii) Let $C([0,1],\mathbb{R})$ be the Banach space consisting of continuous functions from the interval $[0,1]$ to $\mathbb{R}$, equipped with the norm $\|.\|_\infty$. Determine the extreme points of the unit ball of $C([0,1],\mathbb{R})$.