Exam Functional Analysis

KU Leuven, August 26, 2015

Instructions

- You may write your solutions in English or in Dutch. The oral exam is in English or in Dutch, depending on your preference.
- The exam lasts for four hours. You are allowed to eat or drink.
- After one hour, you hand in your solution for Exercise 1. During the rest of the time, you work on the rest of the exercises and you will have your oral exam about Exercise 1.
- The exam is open book. This means that you may use the lecture notes and your own notes. You are not allowed to use anything else.
- The grade for this exam constitutes the whole grade for the reexam. Every of the four exercises has the same weight.
- Write your name on every sheet that you hand in!

Good luck!

Exercise 1.

In this exercise, we consider certain sequences of complex numbers. We use the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

(i) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, and suppose that for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N})$, we have that $(a_n x_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$. Show that $(a_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$.

(ii) Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, and suppose that for every sequence $(y_n)_{n\in\mathbb{N}}$ in $c_0(\mathbb{N})$, we have that $(b_n y_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$. Show that $(b_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$.

Exercise 2.

Let H and K be Hilbert spaces with inner products $\langle .,. \rangle_H$ and $\langle .,. \rangle_K$, respectively. Consider the space $H \oplus K := \{(h,k) \mid h \in H,\, k \in K\}$ (note that as a vector space this is just the Cartesian product of H and K) equipped with the inner product $\langle (h_1,k_1),(h_2,k_2)\rangle = \langle h_1,h_2\rangle_H + \langle k_1,k_2\rangle_K$. Then $H \oplus K$ is a Hilbert space with respect to this inner product (you do **not** have to show this).

Let $A \in B(H)$, $B \in B(K,H)$, $C \in B(H,K)$ and $D \in B(K)$, and define the operator $M\colon H \oplus K \to H \oplus K$ by $M(h,k) = (Ah + Bk, Ch + Dk)$. If we identify the Hilbert space $H \oplus K$ through

$$\left\{ \begin{pmatrix} h \\ k \end{pmatrix} \mid h \in H,\, k \in K \right\},$$

then we can represent the operator M by the matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It is straightforward to verify that M is linear (you do **not** have to show this).

(i) Show that M is a bounded operator.

(ii) Suppose now that $M \in B(H \oplus K)$. Show that the adjoint M^* is given by

$$M^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

(iii) Suppose that $M \in B(H \oplus K)$ and that $D = I_K$ is the identity operator on K. Consider the following two operators:

$$M_1 = \begin{pmatrix} A & B \\ 0 & I_K \end{pmatrix} \qquad \text{and} \qquad M_2 = \begin{pmatrix} A & 0 \\ C & I_K \end{pmatrix}.$$

Show that if A is invertible, then M_1 and M_2 are invertible.

Exercise 3.

Let X be a Banach space, let X^* denote its dual, and let $X^{**} = (X^*)^*$ denote its double dual. Consider the map $\iota\colon X \to X^{**}$ given by $\iota(x)(\omega) = \omega(x)$. This map is an isometry (as explained in Corollary 4.3 of the lecture notes). The Banach space X is said to be reflexive if ι is an isometric isomorphism, i.e. the Banach space X is reflexive if and only if $\iota(X) = X^{**}$.

(i) Show that if X is reflexive, then the unit ball $(X)_1$ of X is compact in the weak topology on X.

(ii) Show that if X is reflexive, then every weakly Cauchy sequence in X converges weakly in X. (Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is called weakly Cauchy if $(\varphi(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence (in the scalar field of X) for every $\varphi \in X^*$.)

Exercise 4.

Let H be a separable infinite-dimensional Hilbert space, and let $B(H)$ be the space of bounded operators on H. In this exercise, we consider three topologies on $B(H)$, namely the norm topology (given by the operator norm on $B(H)$) and the strong topology and weak topology, as introduced in Example 7.4 of the lecture notes. Recall that the strong topology is defined by the family $\mathcal{P}_{\text{strong}}$ of seminorms given by

$$\mathcal{P}_{\text{strong}} = \{T \mapsto \|Tx\| \mid x \in H\},$$

and that the weak topology is defined by the family $\mathcal{P}_{\text{weak}}$ of seminorms given by

$$\mathcal{P}_{\text{weak}} = \{T \mapsto |\langle Tx, y\rangle| \mid x, y \in H\}.$$

(Recall that this topology does not coincide with the weak topology on $B(H)$ when viewing $B(H)$ as a Banach space.)

(i) Show that the map $*\colon B(H) \to B(H)$ given by $T \mapsto T^*$ (where T^* denotes the adjoint operator of T) is continuous when we equip $B(H)$ with the norm topology.

(ii) Show that the map $*\colon B(H) \to B(H)$ given by $T \mapsto T^*$ is continuous when we equip $B(H)$ with the weak topology.

(iii) Show that the map $*\colon B(H) \to B(H)$ given by $T \mapsto T^*$ is **not** continuous when we equip $B(H)$ with the strong topology.

Part of this exercise is part of Exercise 6 of Chapter 7.