Exam Probability & Measure

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Problem 1 (Oral)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function such that $f(0) = 0$. Is the integral $\int_0^\infty \frac{f(x)}{xe^x} \mathrm{d}x$ always finite? If not, construct a counterexample. Can you impose additional natural conditions on f that would imply integrability?

Solution. Let us first say, which points are problematic. One issue is at 0, because the function $\frac{1}{x}$ is not continuous there, and as always we have to worry about infinity. Actually, infinity is not that important, because f is bounded, so there exists a constant $M > 0$ such that $\frac{|f(x)|}{xe^x} \leqslant Me^{-x}$ for $x \geqslant 1$, and this function is clearly integrable.

Let us deal with 0 now, which actually can be a problem. We want to ensure that the integral $\int_0^1 \frac{|f(x)|}{xe^x} \mathrm{d}x$ is finite. It will be more convenient to assume now that f is positive, which we can do. Since e^x is bounded above and below on $[0, 1]$, we only have to deal with $\int_0^1 \frac{f(x)}{x} \mathrm{d}x$. Recall that functions of the form $x \mapsto x^\alpha$ are integrable around zero precisely for $\alpha > -1$. So, as soon as $f(x) \leqslant Cx^\beta$ for some $\beta > 0$ around zero, the integral will be finite. A slightly weaker condition would be differentiability at 0; then $\lim_{x\to 0} \frac{f(x)}{x} = f'(x)$, so this ratio remains bounded.

In order to cook up a counterexample, we need a function that changes more slowly that any power; logarithm is such a function. More specifically, define

$$f(x) := \begin{cases} \frac{1}{\log(\frac{1}{x})} & \text{for } x \leqslant \frac{1}{2} \\ \log(2) & \text{for } x > \frac{1}{2} \end{cases} .$$

And compute $\int_0^{\frac{1}{2}} \frac{f(x)}{x} \mathrm{d}x = \int_0^{\frac{1}{2}} \frac{1}{x\log(\frac{1}{x})} \mathrm{d}x$. By substitution $u = -\log(x)$, we reduce to the integral $\int_{\log(2)}^\infty \frac{1}{u} \mathrm{d}u$, which is infinite.

There is also an abstract way of proving that there must exist a counterexample, but it requires some functional analysis. The first step is employing the closed graph theorem: view $f \mapsto \int_0^1 \frac{f(x)}{x} \mathrm{d}x$ as a linear functional on the Banach space of continuous functions on $[0, 1]$ that vanish at 0, denoted by $C_0((0, 1])$. The closed graph theorem will say that if this functional is everywhere defined, i.e. finite for any such function f, then it must be bounded. But by the Riesz representation theorem (for a locally compact space), any such functional is given by a regular Borel measure with finite variation. But in our case the measure is $\frac{1}{x}\mathrm{d}x$, which is infinite, hence there must exist a counterexample. Another approach would be the following: because of the boundedness of the functional, one does not have to come up with an actual counterexample but only a sequence of approximate counterexample, i.e. a sequence of continuous functions vanishing at 0, bounded 1, but such that the sequence of integrals is unbounded. In order to do that, one may try to approximate the constant function 1 almost everywhere by an increasing sequence of continuous functions vanishing at 0. A specific example might be the following: $f_n = \min(nx, 1)$ – a linear function growing fast on interval $[0, \frac{1}{n}]$ and then constant. $\square$

Problem 2. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $[0,1]$ such that $\int |f_n|^2 \mathrm{d}x = e^{-n}$. Does the series $\sum_{n=1}^{\infty} f_n(x)$ converge for almost every $x \in [0,1]$? Does it converge for every x?

Solution. Since $[0,1]$ equipped with the Lebesgue measure is a probability space, we have by Schwarz inequality $\left(\int |f|\mathrm{d}x\right)^2 \leqslant \int |f|^2 \mathrm{d}x$. It follows that $\int |f_n|\mathrm{d}x \leqslant e^{-\frac{n}{2}}$. By the monotone convergence theorem we have

$$\int_0^1 \sum_{n=1}^{\infty} |f_n|\mathrm{d}x \leqslant \sum_{n=1}^{\infty} e^{-\frac{n}{2}} < \infty.$$

Since the integral of the series is finite, the series must converge almost everywhere. It does not necessarily converge everywhere, because we can redefine all the functions to equal 1 at 0, i.e. $f_n(0) = 1$, without changing the condition on the integrals. But then $\sum_{n=1}^{\infty} f_n(0) = \infty$. $\square$

Problem 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that $\Omega = [0,1]^2$, $\mathcal{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is the Lebesgue measure. Consider the random variables X and Y given by $X(x,y) := x$ and $Y(x,y) = Y$. Compute the conditional expectation $\mathbb{E}(\cos(XY)|Y)$.

Assume now that X_1 and X_2 are independent, real random variables, and that $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a bounded continuous function. Can you find a formula for $\mathbb{E}(g(X_1, X_2)|X_2)$? Verify it using the definition.

Solution. First of all, the result will be a measurable function of Y, so something of the form $g(Y)$. In order to compute g, note that intuitively $g(y) = \mathbb{E}(\cos(XY)|Y = y)$. Since X and Y are independent, any condition on Y does not impose X, so this should be equal to $\int_0^1 \cos(xy)\mathrm{d}x = \frac{\sin(y)}{y}$.

In general, we also need to integrate out the first variable, so the result will by $h(Y)$, where $h(y) := \mathbb{E}g(X, y)$. We will now check that it satisfies the definition. We have to check that $\mathbb{E}h(Y)\mathbb{1}_{Y^{-1}(B)} = \mathbb{E}g(X,Y)\mathbb{1}_{Y^{-1}(B)}$ for any Borel subset $B \subset \mathbb{R}$. Note first that $\mathbb{E}h(Y)\mathbb{1}_{Y^{-1}(B)} = \int_B h(y)\mathrm{d}\mu_Y(y)$. By definition $h(y) = \mathbb{E}g(X,y) = \int_{\mathbb{R}} g(x,y)\mathrm{d}\mu_X(x)$, so we arrive at

$$\int_B (\int_{\mathbb{R}} g(x,y)\mathrm{d}\mu_X(x))\mathrm{d}\mu_Y(y).$$

Since we are dealing with probability measures and g is bounded, we may apply Fubini theorem to obtain $\int_{\mathbb{R}\times B} g(x,y)\mathrm{d}(\mu_X \otimes \mu_Y)(x,y)$. By independence, the product measure is equal to the distribution of (X,Y), so the integral is equal to $\mathbb{E}g(X,Y)\mathbb{1}_{Y^{-1}(B)}$, which is exactly what we wanted. $\square$

Problem 4. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables such that $\mathbb{P}(\varepsilon_n = 0) = \mathbb{P}(\varepsilon_n = 1) = \frac{1}{2}$. Define $X := \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$. Check that X is a random variable and compute its distribution.

Solution. Note that $\sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. It follows that X is a limit of finite sums, which certainly are random variable, so it is a random variable itself.

To compute its distribution, we will use the binary expansion of a real number in $[0,1]$ Note that some numbers have non-unique binary expansion but we can neglect them. Indeed, by Borel-Cantelli lemma, almost surely infinitely many ε_n's are equal to 1, so almost every number we get has infinite expansion. By the same taken, infinitely many ε_n's are equal to 0, so we do not get expansions that have only 1's from some point on. Let us now compute the probability that $X \in \left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)$. Since the length of this interval is equal to $\frac{1}{2^m}$, it means that the values of $\varepsilon_1, \ldots, \varepsilon_m$ are specified so that the sum $\sum_{n=1}^{m} \frac{\varepsilon_n}{2^n} = \frac{k-1}{2^m}$ and we have a complete freedom with other ε_n's. It means that that the measure of $\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right)$ is equal to $\frac{1}{2^m}$. Since these intervals form a semiring that generates the Borel σ-algebra, we conclude that the distribution of X is the Lebesgue measure, i.e. the distribution is uniform.

Another idea is to use the characteristic function. The series defining X converges absolutely, in particular it converges in distribution, so we can compute the characteristic function of X as a limit of characteristic functions of finite sums. Compute first the characteristic function of $\frac{\varepsilon_n}{2^n}$ to get $\frac{1}{2}\left(1 + \exp(\frac{it}{2^n})\right)$. Since these random variables are indepedent, the characteristic function of the sum is equal to the product of characteristic functions, hence the characteristic function of $\sum_{n=1}^{m} \frac{\varepsilon_n}{2^n}$ is equal to $\prod_{n=1}^{m} \frac{1}{2}\left(1 + \exp(\frac{it}{2^n})\right)$. Note that $1 + \exp(\frac{it}{2^n}) = \exp(\frac{it}{2^{n+1}})(\exp(-\frac{it}{2^{n+1}}) + \exp(\frac{it}{2^{n+1}})) = 2\exp(\frac{it}{2^{n+1}})\cos(\frac{t}{2^{n+1}})$. Therefore we end up with the product

$$\prod_{n=1}^{m} \exp(\frac{it}{2^{n+1}})\cos(\frac{t}{2^{n+1}}).$$

The exponential gives a geometric series, which we can sum easily and in the limit it yields $\exp(\frac{it}{2})$; we have to deal with the cosines. Multiply the product $\prod_{n=1}^{m} \cos(\frac{t}{2^{n+1}})$ by $\sin(\frac{t}{2^{m+1}})$ and use the formula $\sin(x)\cos(x) = \frac{\sin(2x)}{2}$. Note that you get $\sin(\frac{t}{2^m})$, so you can pair it $\cos(\frac{t}{2^m})$. At every step you obtain a sine with doubled argument and you divide by 2. The result is $\frac{1}{2^m}\sin(\frac{t}{2})$. Recall that we multiplied by $\sin(\frac{t}{2^{m+1}})$, so we get

$$\prod_{n=1}^{m} \cos(\frac{t}{2^{n+1}}) = \frac{\sin(\frac{t}{2})}{2^m \sin(\frac{t}{2^{m+1}})}.$$

The limit of this expression is equal to $\frac{2\sin(\frac{t}{2})}{t}$, therefore the characteristic function of X is equal to $\frac{2\exp(\frac{it}{2})\sin(\frac{t}{2})}{t}$; we can simplify it to $\frac{e^{it}-1}{it}$. Note that it is precisely the characteristic function of the uniform distribution.

$\square$

Good luck!