Exam Statistical Mechanics

13^{th} January 2020

1 Oral part with written preparation

1.1 Drift-diffusion equation

The drift-diffusion equation describes the time evolution of the particles concentration $c(x,t)$ and is given by

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2} + \frac{1}{\gamma}\frac{\partial}{\partial x}\left[c(x,t)\frac{dV}{dx}\right] \tag{1}$$

Explain the meaning of the different terms of this equation. Why is it called the drift-diffusion equation? Show how one can derive the Einstein relation from it.

1.2 Density of bosons

We have derived the following formula

$$n\lambda_T^3 = \frac{\lambda_T^3}{V}\frac{z}{1-z} + \sum_{l=1}^{+\infty}\frac{z^l}{l^{3/2}} \tag{2}$$

where $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$ is the thermal wavelength, $z = e^{\beta\mu}$ and $\beta = 1/k_B T$. Discuss and analyse this formula for:

1. Low densities and high temperatures such that $n\lambda_T^3 < 1$.
2. High denisties and low temperatures such that $n\lambda_T^3 >> 1$.

Pay particular attention to the ground state and discuss some physical systems where this analysis is applicable and important.

2 Written 1: Classical statistical mechanics

2.1 Canonical and Grand-canonical ensemble *(6 points)*

Consider a system of non-interacting molecules and assume that the one-molecule partition function $Z_1(V,T)$ is known. (This partition function will be obtained by integration of some internal degrees of freedom of the molecules - we do not consider this step and assume Z_1 to be known; our results will depend on Z_1.)

1. Find the total canonical partition function $Z(N,V,T)$ for a system of N molecules.

2. Compute the grand-canonical partition function $\Xi(\mu,V,T)$.

3. Calculate the variance of the number of molecules in the grand-canonical ensemble and show that the relative fluctuations in N vanish in the thermodynamic limit.

2.2 Two-dimensional harmonic oscillator *(8 points)*

We consider a two-dimensional harmonic oscillator of frequency ω. In polar coordinates the hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2m}(p_r^2 + \frac{p_\phi^2}{r^2}) + \frac{m\omega^2 r^2}{2} \tag{3}$$

1. Calculate the canonical partition function for this system using polar coordinates. In this case we integrate in $dp_r dr dp_\phi d\phi$.

2. From the partition function in 1) calculate the internal energy and show that this is consistent with the equipartition theorem.

3. Calculate $< p_\phi^2 >$.

2.3 Relativistic gas *(6 points)*

Consider a system of N relativistic particles in a volume V and at a temperature T. In the limit of small masses the hamiltonian is given by

$$\mathcal{H} = c|\vec{p}| \tag{4}$$

1. Compute the canonical partition function for this sytem ans derive the energy and specific heat. Show that the result is consistent with the equipartition theorem.

2. Determine the pressure as a function of volume, tempereatire and number of particles.

3 Written 2: Phase transistions and quantum statistical mechanics

3.1 Two-dimensional Ising model *(5 points)*

Given that the *exact* sponatneous magnetisation of the two-dimensional Ising model is given by

$$m_0 = [1 - sinh^{-4}\Big(\frac{2J}{k_B T}\Big)]^{1/8}. \tag{5}$$

Calculate T_c the critical temperature of the system. Find $m_0(T)$ to the lowest order in $T - T_c$, e.g. the behaviour of the spontaneous magnetisation around the vicinity of the critical point.

3.2 Energy and specific heat at low temperature *(5 points)*

Consider a statistical system governed by quantum mechanics. Assume that the system is in equilibrium at verly low temperature T. Write down the partition function of the system in the canonical ensemble using an approximation in whch the first two state with energies ϵ_1 ans $\epsilon_2 = \epsilon_1 + \delta\epsilon$ constribute. Compute the average energy E in this approximation. Compute the specific heat, $c_v = \frac{\partial E}{\partial T}$, and discuss its behaviour at low temperature.

3.3 Ideal Fermi gas *(10 points)*

What is the pressure of a gas of free bosons in the limit of vanishing temperature, $T \to 0$? Argue that for $T \to 0$ an ideal Fermi gas will have non-vanishing pressure $p_0 > 0$. We will now use this fact to study a system of two ideal Fermi gases in 3D.
A free sliding piston seperates two compartments labeled 1 and 2 with volumes V_1 and V_2 respectively. An ideal Fermi gas of N_1 particles with mass m_1 and spin $s_1 = 1/2$ is placed in compartment 1 while an ideal Fermi gas of N_2 particles with mass m_2 and spin $s_2 = 3/2$ is placed in compartment 2.

1. Find the density of states $g_{1/2}(\epsilon)$ and $g_{3/2}(\epsilon)$ of the two gases.[1]

[1]Hint: a particle of spin s has $2s + 1$ possible orientations.

2. Find the pressure of the two gases as a function of their densities N_1/V_1 and N_2/V_2 in the limit $T \to 0$.

3. Find the relative densities of the two gaes at mechanical equilibrium in the limit $T \to 0$.

4. What are the equilibrium densities in the classical limit $T \to \infty$?