Exam Statistical Mechanics

22 januari 2018

1 Oral Part

1.1 Ising Model

In the mean field approximation we have the equation

$$m = \tanh(\beta(mJz + H).$$

Explain the meaning of every quantity in the formula. Show that there is a critical temperature so that there is a spontaneous magnetisation for temperatures lower than the critical temperature. In the vicinity of thecritical point, many things behave as a power law. Show this for one or two examples.

1.2 Bosons

We have derived the following formula

$$n\lambda_T^3 = \frac{\lambda_T^3}{V}\frac{z}{1-z} + \sum_{l=1}^{\infty}\frac{z^l}{l^{3/2}}.$$

Discuss and analyse this formula for

1. Low densities and high temperatures such that $n\lambda_T^3 < 1$.
2. High denisties and low temperatures such that $n\lambda_T^3 >> 1$.

Pay particular attention to the ground state and discuss some physical systems where this analysis is applicable and important.

2 Written 1 : Classical

2.1 Chain of oscillators

We consider $N-1$ coupled oscillators such that

$$\mathcal{H} = \sum_{i=1}^{N}\frac{p_i^2}{2m} + \sum_{i=1}^{N-1}\frac{K}{2}(x_{i+1}-x_i)^2.$$

1. Calculate $\langle E\rangle$ and $\langle E^2\rangle - \langle E\rangle^2$ and show that relative fluctuations are small.
2. Repeat for anharmonic oscillators

$$\mathcal{H} = \sum_{i=1}^{N}\frac{p_i^2}{2m} + \sum_{i=1}^{N-1}\frac{K}{2}(x_{i+1}-x_i)^4.$$

2.2 Fluctuations in an ideal gas

We have N particles in a volume V at a temperature T. Consider a part of this volume V_1 with N_1 particles in it.

1. Calculate $\langle N_1 \rangle$ and $\langle N_1^2 \rangle$.
2. Calculate the variance and show that relative fluctuations are small

2.3 2D oscillator

Consider a 2D oscillator in polar coordinates

$$\mathcal{H} = \frac{1}{2m}(p_r^2 + \frac{p_\phi^2}{r^2}) + \frac{m\omega^2 r^2}{2}.$$

1. Calculate the partition function in polar coordinates.
2. Calculate $\langle E \rangle$ and compare with the result in Cartesian coordinates.
3. Do this calculation with the equipartition theorem in polar coordinates.

3 Written 2 : Quantum

3.1 Low temperature limit

Suppose we have a system in equilibrium. We can approximate this in the low temperature by only considering the lowest two states ε_1 and $\varepsilon_2 = \varepsilon_1 + \delta\varepsilon$. Calculate $\langle E \rangle$ and $c_V = \frac{\partial E}{\partial T}$ and determine what happens in the limit $T \to 0$.

4 Black body

We know that the energy distribution for a black body is given by

$$\epsilon_\omega d\omega = \frac{\hbar^2}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{\hbar\omega} - 1}.$$

1. Use $\lambda\omega = 2\pi c$ to find $\epsilon_\lambda d\lambda = \epsilon_\omega d\omega$.
2. Search for the maximum of the distribution ω_{max} and λ_{max}. Use the approximate solutions $x - 3(1 - e^{-x}) = 0$ for $x = 2,827$ and $x - 5(1 - e^{-x} = 0$ for $x = 4,966$.
3. If for the sun $\lambda_{max} = 5 \cdot 10^{-7} m$ estimate the temperature of the sun.
4. Now estimate the temperature of the sun. We have $\varepsilon = \frac{E}{V}$. Calculate this, paying particular attention to the temperature dependence. Use now the formula $\epsilon_S = \frac{4d^2}{R_S^2}\epsilon_E$ to calculate the temperature of the sun. d is the distance between the Sun and the Earth.