Exam: Spectral Theory and Operator Algebras

June 14 2019

1. Let X be a locally compact Hausdorff topological space. Let $C_b(X)$ be the unital C^*-algebra of all continuous and bounded functions $X \to \mathbb{C}$. Take $f \in C_b(X)$, show that the spectrum is given by $\sigma(f) = \overline{f(X)}$.

2. Let A be unital C^*-algebra. Observe the assignment $p \mapsto 2p - 1$ from the projections $p \in A$ to the self adjoint unitaries of A. Show that this is a bijection. Deduce the inverse map.

3. Let A be a non unital C^*-algebra and ϕ a state. Let a_λ be a net in A such that $a_\lambda b \to 0$ for all $b \in A$. Show that $\phi(a_\lambda) \to 0$.

4. Let $A \subseteq \mathcal{L}(H)$ be a concrete C^*-algebra and $M = A''$. Show that for every unitary u in M, there exists a net of unitaries v_λ in A, such that $v_\lambda \to u$ in the strong operator topology.

   Hint: Remember that $\exp(it) = \cos(t) + i\sin(t)$ for all $t \in \mathbb{R}$.

5. Let $S \in \mathcal{L}(l^2(\mathbb{N}))$ be the one-sided shift operator given by $S(x_1, x_2 x_{,3}, ...) = (0, x_1, x_2, ...)$. Let $\mathcal{T}$ be the Toeplitz algebra with $t \in \mathcal{T}$ its universal isometry.

   (a) Show that there exists a representation $\pi : \mathcal{T} \to \mathcal{L}(l^2(\mathbb{N}))$ such that $\pi(t) = s$.

   (b) Show that π is irreducible.

   Hint: What happens when π is restricted to the ideal generated by $\mathbf{1} - tt^*$?

6. Let $\epsilon \in (0, 1)$ be a number.

   (a) Justify the existence of the universal C^*-algebra $B_\epsilon = C^*(s \mid \|ss^* - \mathbf{1}\| \leq \epsilon)$.

   (b) Show that the Toeplitz algebra $\mathcal{T}$ is isomorphic to a quotient of B_ϵ.

   (c) Show there exists a unital embedding $\mathcal{T} \hookrightarrow B_\epsilon$.