Exam 09/01/2021 Groups and symmetries

Student

January 2021

1 Theoretical part

Closed book; Written; 30 min

1. Given a Lie algebra with generators L^i which have the commutation relations $[L^i, L^j] = i\epsilon^{ijk}L^k$. Is this a real algebra? How can you tell? How can you make this into a real algebra?
2. Suppose a lie algebra can be split as $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$. What has to be the structure of commutators such that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.
3. Describe a root as an element of a dual vector space. Which commutation relation defines these linear maps. These are also solutions of an equation. Which one?
4. Often we work which the components of these roots α^i. In which commutation relations do they appear? How are the components labeled by i defined in the Chevally basis?
5. The Cartan-Weyl basis defines a real form. How is it called? Is it compact? What is its character?
6. What is the formal mathematics definition of a Lie group? For this, two essential structures are needed and a compatibility condition.
7. How can you recognize the fundamental Weyl domain? What is its content for representation theory?
8. Suppose you have tensor A^{ijk} where $A^{ijk} = -A^{jik}$ and $A^{ijk} + A^{kij} + A^{jki} = 0$. What is the corresponding Young tableaux?

2 Practical part

Open book; Written; 2 h 30 min

2.1 Question 1: Lorentz algebra

Suppose the lie algebra $\mathfrak{so}(1,3)$, i.e. Lorentz algebra. It has the generators

$$M_{\mu\nu} = g_{\mu\rho}M^{\rho}{}_{\nu} = -M_{\nu\mu} \tag{1}$$

where

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_3 \end{pmatrix}; \qquad g_{00} = -1; \qquad g_{0i} = g_{i0} = 0, \qquad g_{ij} = \delta_{ij}. \tag{2}$$

The commutation relation are given by

$$\left[M_{\mu\nu}, M_{\sigma\rho}\right] = g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\rho}M_{\nu\sigma} \tag{3}$$

(a) There are 6 generators. Can they all 6 be hermitian? Can they all 6 be anti-hermitian?

(b) Let us define the generator $L^i := \frac{1}{2}\epsilon^{ijk}M_{jk}$. Prove that the commutator is given by $\left[L^i, L^j\right] = -\epsilon^{ijk}L^k$.

(c) Define $K^i := M^i{}_0$. Prove that the following commutators hold

$$\left[K^i, K^j\right] = \epsilon^{ijk} L^k; \qquad \left[L^i, K^j\right] = -\epsilon_{ijk} K^k. \tag{4}$$

(d) According to the commutations relation, can one realize K^i as anti-hermitian or hermitian generators?

(e) Calculate the Cartan-Killing metric.

(f) If we look at the killing form as the trace should K^i be hermitian or anti-hermitian? Explain with the positivity of the trace.

(g) We have seen that the Lorentz algebra can be realized as a real algebra. Which algebra is this? Explain why.

(h) Prove that algebra realized by linear combination of L^i, K^i, i.e. $a_i L^i + b_i K^i$ can also be realized by $(a_i + ib_i)L^i$. Which algebra is this?

(i) What changes if we set g the euclidean metric, i.e. $g_{00} = 1$?

(j) Define the algebra $J^i_\pm = \frac{1}{2}(L^i \pm K^i)$. What do you find? Which equivalence of Lie algebras do you find?

2.2 Question 2: Nahm result

There was a page-long explanation. Alas, it has been lost. It was something about bosonic subgroups and fermionic subgroups and supergroups.

An attempt at reconstruction: The symmetry group of the de Sitter space is $SO(D,1)$, and the one of the anti-de Sitter space is $SO(D-1,2)$. The groups of physical interest are the so-called "Lie supergroups" for which the "bosonic subgroups" have one of these groups as a tensor factor. Then the other factor is called the R-symmetry group. A table was given with some "bosonic subgroups", which were decomposed as tensor products of known Lie groups (e.g. $SO(n-p,p)$, $SU^*(n)$, $Usp(m)$, etc, and also $\overline{SO(7-p,p)}$).

Actually, we don't want the group $SO(D,1)$ and $SO(D-1,2)$ itself, but the universal covering group. As seen in the lecture, $SU(2)$ is the universal covering group of $SO(3)$. For some other dimensions, we also found some named groups for the universal covering group. Assume we can do something similar for e.g. $SO(2,1)$.

Now classify all physically relevant supergroups for $D \geq 4$. Identify the R-symmetry groups. Do you see something with regards to compactness of the R-symmetry groups?

The way I understand it, the main goal of the exercise was to look at table IX, page 141 in Fuchs & Schweigert and to "lift" the isomorphisms there to isomorphisms between the universal covering group of the relevant $SO(n,m)$ and the groups with the name of the isomorphic algebras listed in the table.