Groups & Symmetries: Practical Part Exam

January 21, 2022

Real forms

In the lecture, we defined the character of real forms. Calculate the character of the following real forms:

1. $\mathfrak{su}(p,q)$;
2. $\mathfrak{su}^*(2n)$;
3. $\mathfrak{sl}(n)$;
4. $\mathfrak{so}^*(2n)$;
5. $G_{2,2}$.

Galilei group

The Galilei group G is the invariance group of classical non relativistic mechanics. The coordinates of space are indicated by x^i, where $i = 1, 2, 3$ and there is a time coordinate t. The generators can be written as

$$J_i = \epsilon_{ijk} x^j \frac{\partial}{\partial x^k}\,, \quad P_i = \frac{\partial}{\partial x^i}\,, \quad K_i = t\frac{\partial}{\partial x^i}\,, \quad H = \frac{\partial}{\partial t}\,.$$

The commutators are easy to compute. I will save you the work. One gets

$$\begin{aligned}
&[J_i, J_j] = \epsilon_{ijk} J_k\,, && [J_i, P_j] = -\epsilon_{ijk} P_k\,,\\
&[J_i, K_j] = -\epsilon_{ijk} K_k\,, && [J_i, H] = 0\,,\\
&[P_i, H] = 0\,, && [P_i, P_j] = 0\,,\\
&[K_i, H] = -P_i\,, && [K_i, K_j] = 0\,,\\
&[P_i, K_j] = 0\,.
\end{aligned}$$

1. Which generators are in the derived algebra?
2. Is G a solvable algebra?
3. Can the previous steps tell you whether G is a simple algebra?

4. Do you recognize a simple subalgebra? Which name has it in the $A, B, \dots, E$ classification of algebras?

5. Can you then write down what is the Levi decomposition of the algebra? Does each part satisfy what it should satisfy for a correct Levi-decomposition?

Representation of $\mathfrak{su}(3)$

Consider a representation of $\mathfrak{su}(3)$, where the weight vectors have Dynkin labels

$$\begin{array}{cccccc} (3,-1)\,, & (2,1)\,, & (2,-2)\,, & (1,0)\,, & (1,-3)\,, & (0,2)\,, \\ (0,-1)\,, & (-1,1)\,, & (-1,-2)\,, & (-2,3)\,, & (-2,0)\,, & (-3,2)\,. \end{array}$$

1. What is the highest weight?

2. Is it a self-conjugate representation? If so, why? If no, what is the conjugate representation (Dynkin label)?

3. Draw the Young tableaux that corresponds to this representation.

4. We can obtain the representation from the product of 2 representations: one is a completely symmetric tensor T_{ijk} and the other one a vector V_i ($i = 1, 2, 3$). Thus we consider $U_{ijkl} = T_{ijk}V_l$. Which further contraint should we impose on U_{ijkl} to obtain the representation that we discussed?