Relativity

(01/02/2013 (9u00-13u30))

1 (a) Show that the Weyl tensor $C^{\mu}{}_{\nu\rho\sigma}$ is left invariant by a conformal transformation.

(b) Show that any Killing vector satisfies

$$\nabla_\mu \nabla_\rho K^\mu = R_{\sigma\nu} K^\nu$$
$$K^\lambda \nabla_\lambda R = 0$$

Deduce that in Minkowski space-time the components of Killing covectors are linear functions of the coordinates.

2 (a) Let E denote the maximum energy that can be extracted from a Kerr black hole in the Penrose process. The efficiency of this process is $\eta \equiv E/M$ where M is the initial mass of the black hole. Calculate what is the largest possible value of η.

(b) Evaluate the Komar integral for the angular momentum of an asymptotically flat axisymmetric space-time tot verify that $J = Ma$ for the Kerr black hole with rotation parameter a.

(c) The specific heat of a charged black hole of mass M, at fixed charge Q, is

$$C \equiv T_{\mathrm{H}} \left. \frac{\partial S_{\mathrm{BH}}}{\partial T_{\mathrm{H}}} \right|_Q ,$$

where T_{H} is its Hawking temperature and $S_{\mathrm{BH}} = \frac{1}{4}A$, where A is the area of the event horizon, show that the specific heat of a Reissner-Nordström black hole is

$$C = \frac{2S_{\mathrm{BH}}\sqrt{M^2 - Q^2}}{M - 2\sqrt{M^2 - Q^2}}.$$

Is there a critical value of the mass at which C^{-1} changes sign?

3 The generalization of the Schwarzschild solution to d dimensional space-time is the Schwarzschild-Tangherlini metric

$$\mathrm{d}s^2 = -\left[1 - \left(\frac{a}{r}\right)^{d-3}\right]\mathrm{d}t^2 + \left[1 - \left(\frac{a}{r}\right)^{d-3}\right]^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{d-2}^2,$$

where $a > 0$ is constant and $\mathrm{d}\Omega_{d-2}^2$ is the standard round metric on S^{d-2}, defined inductively bij $\mathrm{d}\Omega_1^2 = \mathrm{d}\varphi^2$ and $\mathrm{d}\Omega_{i+1}^2 = \mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\Omega_i^2$ for $i \geq 1$, with $0 < \varphi \leq 2\pi$ and $0 \leq \theta_i \leq \pi$. Show that there are no stable bounded massive orbits for $d \geq 5$. Why might you have expected this result?

4 Consider de Sitter space in static coordinates

$$\mathrm{d}s^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.$$

(a) Identify the Killing horizon. What is its surface gravity? Does the Killing horizon have a nonzero temperature?

(b) Draw the horizon on a conformal diagram. Do the static coordinates cover the entire space-time? What is the significance of the horizon from the viewpoint of a single observer in de Sitter space?

$\boxed{5}$ (a) Consider the Robertson-Walker universe that best fits the current observations, with density parameters $\Omega_{R0} = 10^{-4}$, $\Omega_{M0} = 0.3$ and $\Omega_{\Lambda 0} = 0.7$.
Sketch the behavior of $\rho_{\text{crit}}\Omega$ for the three Ω's as a function of the scale factor a on a log scale from $a = 10^{-35}$ to $a = 10^{35}$. Indicate the Planck time, nucleosynthesis and today.

(b) Sketch the evolution of the scale factor in the same universe. Indicate the period during which large-scale structures such as galaxies form. What would the universe have been like if Λ had been significantly larger?