Take home exam Functional Analysis

Project 1. Hölder inequality for operators.

- Read carefully the instructions on Toledo, including the due date for the solutions.
- If something is unclear, please feel free to ask me, during the lecture or by e-mailing me at stefaan.vaes@wis.kuleuven.be.
- If you cannot prove one of the statements in a question, continue to the next question and use the non-proven statement as a black box.

Study yourself sections 3.5 and 3.6 in the lecture notes. You may use the results from these sections without proving them. In this take home exam you only need the square root of a *compact* positive operator, i.e. the version of Theorem 3.14 that is really proved in the notes.

Throughout this take home exam, H denotes a separable Hilbert space.

1. Let $S \in \mathcal{K}(H)^+$ be a compact positive operator. Prove that for every $\lambda > 0$,

$$\operatorname{span}\{\xi \mid \xi \in H \text{ is an eigenvector with eigenvalue} \geq \lambda \}$$

is finite dimensional.

Conclude that S admits an orthonormal family of eigenvectors $(e_k)_{k=0}^n$ (with $n = +\infty$ or $n < +\infty$) such that

- the corresponding eigenvalues (λ_k) are strictly positive;
- we have $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$;
- we have $S\xi = 0$ if $\xi \perp e_k$ for all k.

We use the notation $s_k(S) := \lambda_k$. If $n < +\infty$, we put by convention $s_k(S) = 0$ for all $k > n$.

For an arbitrary compact operator $S \in \mathcal{K}(H)$ define $s_k(S) := s_k(|S|)$.

2. Use the polar decomposition of a bounded operator to prove that $s_k(S^*) = s_k(S)$ for all $k \in \mathbb{N}$ and $S \in \mathcal{K}(H)$.

3. Prove that $s_0(S) = \|S\|$ and that

$$s_k(S) = \inf\{ \|S_{|K^\perp}\| \mid K \subset H \text{ is a } k\text{-dimensional subspace} \}$$

for all $S \in \mathcal{K}(H)$ and $k \in \mathbb{N}$.

Deduce that for all $S \in \mathcal{K}(H)$, $T \in \mathrm{B}(H)$ and $k \in \mathbb{N}$ we have

$$s_k(TS) \leq \|T\| \, s_k(S) \quad \text{and} \quad s_k(ST) \leq \|T\| \, s_k(S) \, .$$

4. Whenever $S \in \mathcal{K}(H)$ and $1 \le p < \infty$, denote

$$\|S\|_p := \Big(\sum_{k=0}^{\infty} s_k(S)^p\Big)^{1/p} .$$

Prove that $\|S\|_1 = \mathrm{Tr}(|S|)$ and $\|S\|_2 = \sqrt{\mathrm{Tr}(S^*S)}$. So the above notation $\|\cdot\|_p$ is compatible with the notations $\|\cdot\|_1$ and $\|\cdot\|_2$ in the lecture notes.

5. Compute $\|S\|_p$ when S is a multiplication operator on $\ell^2(\mathbb{N})$.

6. Prove that for every compact operator S and all orthonormal families (η_n) and (μ_n) in H we have

$$\sum_{n=0}^{\infty} |\langle S\eta_n, \mu_n\rangle|^p \le \|S\|_p^p .$$

Hint. Take the polar decomposition $S = V\,|S|$. Prove that for a good choice of orthonormal family (e_k) we have

$$|\langle S\eta, \mu\rangle| \le \sum_{k=0}^{\infty} s_k(S)\; |\langle \eta, e_k\rangle|\; |\langle e_k, V^*\mu\rangle|$$

for all $\eta, \mu \in H$. Then use the convexity of $x \mapsto x^p$.

7. Prove that for all compact operators $S, T \in \mathcal{K}(H)$ and all $1 < p, q < \infty$ with $1/p + 1/q = 1$, we have

$$\|ST\|_1 \le \|S\|_p\, \|T\|_q .$$

So Hölder's inequality holds for operators.
Hint. Diagonalize $|T|$, use the ordinary Hölder inequality and use the previous exercise.

8. Use the Hölder inequality to prove that for all compact operators $S, T \in \mathcal{K}(H)$ and all $p \ge 1$, we have

$$\|S + T\|_p \le \|S\|_p + \|T\|_p .$$

9. Let $1 < p < \infty$. Define $\mathcal{L}^p(H) := \{S \in \mathcal{K}(H) \mid \|S\|_p < \infty\}$. Prove that $\mathcal{L}^p(H)$ is a vector space and that $\|\cdot\|_p$ is a norm on this vector space that turns $\mathcal{L}^p(H)$ into a Banach space that can be isometrically identified with the dual of $\mathcal{L}^q(H)$, where $1 < q < \infty$ is chosen such that $1/p + 1/q = 1$.