Take home exam Functional Analysis

Project 3. Amenable actions of countable groups

- Read carefully the instructions on Toledo, including the due date for the solutions.
- If something is unclear, please feel free to ask me by e-mailing me at stefaan.vaes@wis.kuleuven.be.
- If you cannot prove or solve (part of) a question, continue to the next question and use the non-proven statement as a black box if needed.

This project makes use of the following concepts.

- An *action of a group G on a set X* is a map

$$G \times X \to X : (g, x) \mapsto g \cdot x$$

  satisfying $e \cdot x = x$ for all $x \in X$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and all $x \in X$. An action of G on X is actually nothing else than a group homomorphism from G to the group of all permutations of X.

- An action of a countable group G on a countable set X is said to be *amenable* if X admits a mean m (in the sense of Definition 7.7) that satisfies

$$m(g \cdot A) = m(A) \quad \text{for all } \; g \in G, A \subset X \; .$$

  A countable group G acts on itself by left multiplication. It follows directly from the definitions that this left multiplication action is amenable if and only if the group G is amenable.

- An action of a countable group G on a countable set X is said to be
  - *transitive,* if $X = G \cdot x$ for some $x \in X$ (and then, $X = G \cdot x$ holds for all $x \in X$),
  - *faithful,* if the neutral element $e \in G$ is the only $g \in G$ that satisfies $g \cdot x = x$ for all $x \in X$. Note that faithfulness is much weaker than *freeness* where one requires that $g \cdot x \neq x$ for all $g \neq e$ and all $x \in X$.

- A countable group G is said to be *residually finite* if for every $g \in G$ with $g \neq e$ there exists a group homomorphism $\pi : G \to G_0$ such that G_0 is a finite group and $\pi(g) \neq e$.

1. Prove that an action of a countable group G on a countable set X is amenable if and only if there exists a sequence of finitely supported functions $\xi_n : X \to [0, +\infty)$ satisfying $\|\xi_n\|_1 = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|\xi_n \cdot g - \xi_n\|_1 = 0$ for all $g \in G$.

   Here we use the notation $\xi \cdot g$ to denote the function on X given by $(\xi \cdot g)(x) = \xi(g \cdot x)$ for all $x \in X$, given a function ξ on X and a group element $g \in G$.

2. Let G be an amenable group. Prove that every action of G on a countable set is amenable.

3. Study exercise 2 at the end of lecture 9. You do not have to hand in the solution of this exercise. Then prove that an action of a countable group G on a countable set X is amenable if and only if there exists a sequence of finite nonempty subsets $A_n \subset X$ such that

$$\lim_{n\to\infty} \frac{|g \cdot A_n \triangle A_n|}{|A_n|} = 0 \quad \text{for all } g \in G .$$

A sequence of finite subsets $A_n \subset X$ with the above property is called a Følner sequence.

For quite a while it was an open question whether nonamenable groups could admit amenable actions that are faithful and transitive. It is instructive, but not part of the take home exam, to convince yourself that faithfulness and transitivity are added to avoid trivialities. The final aim of this project is to construct an amenable, faithful, transitive action of the free group $\mathbb{F}_2$.

4. Let X be a set and α, β permutations of X. Denote by a, b the generators of the free group $\mathbb{F}_2$. Prove that there is a unique action of $\mathbb{F}_2$ on X such that $a \cdot x = \alpha(x)$ and $b \cdot x = \beta(x)$ for all $x \in X$.

5. For every integer $n \neq 0$ consider the ring $\mathbb{Z}/n\mathbb{Z}$ and consider the group $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. Prove that $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is a finite group that is a quotient of $\mathrm{SL}_2(\mathbb{Z})$. Use Proposition 9.17 to deduce that $\mathbb{F}_2$ is a residually finite group.

6. Choose for all $n \in \mathbb{Z}$ finite groups G_n and surjective homomorphisms $\pi_n : \mathbb{F}_2 \to G_n$ such that for all $g \in \mathbb{F}_2, g \neq e$, we have $\pi_n(g) \neq e$ if $|n|$ is large enough. Make your choice such that $G_0 = \{e\}$ and $|G_n| \to +\infty$ if $|n| \to +\infty$.

   Define the countable set X as the disjoint union of the finite sets G_n. View the G_n as finite subsets of X. To avoid ambiguity, denote by e_n the neutral element of G_n and view as such e_n as an element of X.

   Denote by a, b the canonical generators of $\mathbb{F}_2$. Define the permutations α, β of X by the formulae

$$\alpha(x) = \pi_n(a)x \ \text{ if } \ x \in G_n \quad , \quad \beta(x) = \begin{cases} \pi_n(b)x \in G_n & \text{if } \ x \in G_n \text{ and } x \neq e_n \ , \\ \pi_{n+1}(b) \in G_{n+1} & \text{if } \ x = e_n \ . \end{cases}$$

   Consider the unique action of $\mathbb{F}_2$ on X such that $a \cdot x = \alpha(x)$ and $b \cdot x = \beta(x)$ for all $x \in X$.

   1. Prove that the subsets $G_n \subset X$ form a Følner sequence for the above action of $\mathbb{F}_2$ on X.
   2. Prove that the above action of $\mathbb{F}_2$ on X is faithful by proving that for all $g \in \mathbb{F}_2$ with $g \neq e$ we have that $g \cdot e_n \neq e_n$ whenever n is large enough.
   3. Finally prove that the above action of $\mathbb{F}_2$ on X is transitive. For this the following hints will be useful. Denote by $c_n \geq 1$ the order of $\pi_n(b)$ in the finite group G_n. Prove that $b^{c_n} \cdot e_{n-1} = e_n$ for all $n \in \mathbb{Z}$. Deduce that $e_n \in \mathbb{F}_2 \cdot e_0$ for all $n \in \mathbb{Z}$. Write $H_n := G_n \cap \mathbb{F}_2 \cdot e_0$. Prove that $e_n \in H_n$, that $\pi_n(a)H_n = H_n$ and that $\pi_n(b)H_n = H_n$. Deduce that $H_n = G_n$ and conclude.

   So the above action of $\mathbb{F}_2$ is amenable, faithful and transitive!