Functional Analysis Exam

31 January 2019

1 Exercise 1 (Oral)

Let X be a seminormed space. Let $Y \subset X$ be a vector subspace and $x_0 \in X$. Prove that the following statements are equivalent.

1. x_0 belongs to the closure of Y.
2. For every continuous linear map $\omega : X \to \mathbb{C}$ with $Y \subset \ker \omega$ we have $\omega(x_0) = 0$.

2 Exercise 2

Let X be a Banach space and Y a vector subspace. Y is complemented if Y is closed and there exists a closed vector space $Z \subset X$ satisfying $Y \cap Z = \{0\}$ and $X = Y + Z$. Prove that the following statements are equivalent :

1. Y is complemented
2. There exists a bounded linear map $E : X \to X$ satisfying $E \circ E = E$ and $im(E) = Y$.

(Hint : use the closed graph theorem)

3 Exercise 3

Consider the set $C \subset \ell^2(\mathbb{N})$ defined by

$$C = \{x \in \ell^2(\mathbb{N}| \sum_{n=0}^{\infty}(n+1)x(n) = 1, x(n) \geq 0 \forall n \in \mathbb{N}\}$$

1. Prove that C is convex and the extreme points of C are given by

$$\delta_n(m) = \begin{cases} \frac{1}{n+1} \text{ if } m = n \\ 0 \text{else} \end{cases}$$

2. Is C closed? Describe the closure $\overline{C}$ of C. What are the extreme points of $\overline{C}$?
3. The set $K = \{x \in l^2(\mathbb{N})||x(n)| \leq \frac{1}{n+1}\}$ is the Hilbert cube. It is well known that K is compact and you may use this. What can you say about the closure of $conv(ext(\overline{C}))$? Write down explicitly any point of $\overline{C}$ as a limit of convex combinations of the extreme points of $\overline{C}$.

4 Exercise 4

Consider the Banach space $\ell^\infty(\mathbb{N})$ with the supremum norm. Define $K \subset \ell^\infty(\mathbb{N})^*$ given by

$$K = \{\omega \in \ell^\infty(\mathbb{N})^* | \omega(1) = 1, ||\omega|| \leq 1, \omega(FG) = \omega(F)\omega(G) \forall F, G \in \ell^\infty(\mathbb{N})\}.$$

1. Prove that K is compact in the weak* topology.
2. For every $A \subset \mathbb{N}$ consider the indicator function $\chi_A \in \ell^\infty(\mathbb{N})$. Prove that for every $A \subset \mathbb{N}$ and every $\omega \in K$, we have $\omega(\chi_A) \in \{0, 1\}$.
3. Fix $n \in \mathbb{N}$ and define $\delta_n \in K$ by $\delta_n(F) = F(n)$. Prove that the singleton $\{\delta_n\}$ is an open subset of K in the weak* topology. (Hint : use $\chi_{\{n\}}$ and Lemma 10.8)