Exam Groups and symmetries
2018-19

1 General setup

The final grade of the course will be based on a written homework assignment (approx. 40% of the final grade) and a final oral exam (approx. 60% of the final grade). The presentations of the exams takes place on Wednesday 23/1 at 9:00 in on Friday 25/1 at 14:00 in Celestijnenlaan 200S - 00.04.

For the written part, there are 5 exercises to solve. For the second part, each student will get an extra subject, chosen during the last lecture, i.e. Tuesday 11/12, on which they will have to make a presentation.

The students should send me their written answers to the exercises at the latest 4 days before their exam moment (hence 19/1 or 21/1 depending on the exam moment). Furthermore they should send me at the same time also some notes on their extra subject. These notes should contain the details of derivations and calculations to prepare the presentation. One can write a few words to clarify what is done and why. It does not have to be a summary of the presentation. It should not contain everything what the students read, but I should see from the text which calculations were performed. One may use TeX or one may scan handwritten notes if these are written very clearly.

For the final exam the students will prepare an oral presentation of 20 min. on the blackboard. This presentation should explain what they have learned. Thus the explanation should be understandable for someone who followed the course. The students will be allowed to use notes during the oral presentation/exam. After the presentation there will be questions related to: the advanced topic presented; the general content of the course; and the homework assignments.

In the half-day in which they make their presentation, they should also be present in the presentations of the other students, and *actively* follow the talks.

2 Exercises

2.1 Two-dimensional example

Consider the transformations of coordinate of a point on a line. We can

- rescale the basis vector
- choose another base point on the line
- invert the orientation of the basis vector.

In terms of the coordinate of a point these are the transformations:

$$x' = \alpha_1 x + \alpha_2 \tag{1}$$

This is a 2-dimensional group with $\alpha_1 \neq 0$. First, why should we exclude $\alpha_1 = 0$ in order to get a group ?

Find the composition law. Make a second transformation $x' \mapsto x''$ with parameters (β_1, β_2) . Does this lead to a new transformation of the same type ? Which parameters you obtain for the transformations $x''(x)$?

Check the properties that should be satisfied to have a group.

Check the representation of this topological group by matrices

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 1 \end{pmatrix} . \tag{2}$$

Another representation space has as basis $\{x^N, x^{N-1}, \ldots 1\}$ for a chosen N. The transformation $x \to \alpha_1 x + \alpha_2$ determines the transformations of the elements of the basis. Which is the dimension of that representation? Compare with the previous one.

Prove that the

Consider the subgroup $H = \{(1, \alpha_2)\}$ for $\forall \alpha_2$. Prove that this is an invariant subgroup.

Expanding (2), define matrices T^1 and T^2 as generators. Check that they form a non-abelian algebra, and obtain the structure constants. Calculate the matrices of the adjoint representation. Check that they satisfy the same commutation relations.

Calculate the Cartan–Killing metric for our example, and find that it is degenerate. Can you also prove that the metric cannot be nondegenerate for an algebra with only 2 generators?

Hint: consider $f^{abc} \equiv f^{ab}{}_e \kappa^{ec}$.

Finally, consider the exponentiation of the algebra, and prove that this leads again to the same Lie group (after a reparametrization).

2.2 Dynkin diagrams of exceptional E-series

Consider Dynkin diagrams of the form of Figure 1. Denote the simple root vector in the

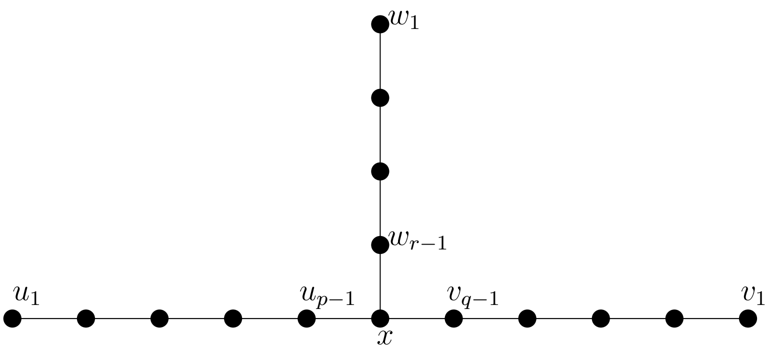


Figure 1: *Three chains connected to a point.*

center, normalized to have length 1, as x. Define similarly such normalized simple roots for the other dots in the diagram, as u_i, v_i, w_i as indicated. Define

$$\vec{u} \equiv \sum_{i=1}^{p-1} i\vec{u}_i, \qquad \vec{v} \equiv \sum_{j=1}^{q-1} j\vec{v}_j, \qquad \vec{w} \equiv \sum_{j=1}^{r-1} j\vec{w}_j. \tag{3}$$

From the Dynkin diagram, we know the inner products. Determine the lengths of these and their inner product with x. When you normalize u, v and w they define an orthogonal frame (why?) of a 3-dimensional subspace. The length of x in this part should be smaller than 1. Determine from this the allowed D and E algebras. Note that you find that E_9 is a limit case. Indeed that infinite-dimensional algebra can be defined in a vector space that has one zero vector.

2.3 Real algebras

Consider the table of isomorphisms between low-dimensional semi-simple Lie algebras; (here $\mathfrak{sp}$ and $\mathfrak{s\ell}$ are the matrices over $\mathbb{R}$ unless mentioned differently):

$$\begin{aligned}
A_1 = B_1 = C_1 : &\mathfrak{so}(3) = \mathfrak{su}(2) = \mathfrak{su}^*(2)\,, \qquad \mathfrak{so}(2,1) = \mathfrak{s\ell}(2) = \mathfrak{su}(1,1) = \mathfrak{sp}(2)\,,\\
D_2 = A_1 + A_1 : &\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)\,, \qquad \mathfrak{so}(3,1) = \mathfrak{sp}(2,\mathbb{C}) = \mathfrak{s\ell}(2,\mathbb{C})\,,\\
&\mathfrak{so}(2,2) = \mathfrak{s\ell}(2) \oplus \mathfrak{s\ell}(2)\,, \qquad \mathfrak{so}^*(4) = \mathfrak{su}(1,1) \oplus \mathfrak{su}(2)\,,\\
B_2 = C_2 : &\mathfrak{so}(5) = \mathfrak{usp}(4)\,, \qquad \mathfrak{so}(4,1) = \mathfrak{usp}(2,2)\,, \qquad \mathfrak{so}(3,2) = \mathfrak{sp}(4)\,,\\
D_3 = A_3 : &\mathfrak{so}(6) = \mathfrak{su}(4)\,, \qquad \mathfrak{so}(5,1) = \mathfrak{su}^*(4)\,, \qquad \mathfrak{so}(4,2) = \mathfrak{su}(2,2)\\
&\mathfrak{so}(3,3) = \mathfrak{s\ell}(4)\,, \qquad \mathfrak{so}^*(6) = \mathfrak{su}(3,1)\,,\\
D_4 : &\mathfrak{so}^*(8) = \mathfrak{so}(6,2)\,.
\end{aligned} \tag{4}$$

It is rather complicated to establish the isomorphisms completely. For each algebra count the number of compact and the number of non-compact generators from the definitions of the algebras.

This table is a small modification and extension from table IX in the book of Fuchs and Schweigert. Note that I denote the symplectic algebras as $\mathfrak{sp}(2n)$, which is $\mathfrak{sp}(n,0)$ in their notation.

You can also use (what we did during the lecture, but is maybe not obvious from the book) the following table of real forms and their maximal compact subalgebras

It turns out that $\mathrm{SU}(N)$ and $\mathrm{USp}(2N)$ are simply connected, but that is not so for $\mathrm{SO}(N)$, as we already saw for $\mathrm{SO}(3)$. Hence, there is in general another group: $\overline{\mathrm{SO}(N)}$, which is simply connected. We already saw that $\mathrm{SU}(2) = \overline{\mathrm{SO}(3)}$. But in general $\overline{\mathrm{SO}(N)}$ is a group that is distinct from another classical group.

With all the information that you already have, you should be able to find two other examples where $\overline{\mathrm{SO}(N)}$ is equal to a group that has another name. Which are these?

Table 1: *Real forms of the classical Lie algebras.*

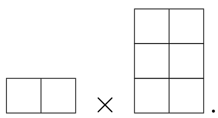
Complex extension	compact form $\mathfrak{g}$	Real form $\mathfrak{g}^*$	Maximal compact subalgebra $\mathfrak{k}$
$\mathfrak{sl}(n;\mathbb{C})$	$\mathfrak{su}(n)$	$\mathfrak{su}(p,n-p)$	$\mathfrak{su}(p)\oplus\mathfrak{su}(n-p)\oplus\mathfrak{u}(1)$
$\mathfrak{sl}(2n;\mathbb{C})$	$\mathfrak{su}(2n)$	$\mathfrak{su}^*(2n)=\mathfrak{s}\ell(n,\mathbb{C})$	$\mathfrak{usp}(2n)=\mathfrak{u}(n,\mathbb{H})$
$\mathfrak{sl}(n;\mathbb{C})$	$\mathfrak{su}(n)$	$\mathfrak{sl}(n;\mathbb{R})$	$\mathfrak{so}(n)$
$\mathfrak{so}(n;\mathbb{C})$	$\mathfrak{so}(n)$	$\mathfrak{so}(p,n-p)$	$\mathfrak{so}(p)\oplus\mathfrak{so}(q)$
$\mathfrak{so}(2n;\mathbb{C})$	$\mathfrak{so}(2n)$	$\mathfrak{so}^*(2n)=\mathfrak{o}(n,\mathbb{H})$	$\mathfrak{u}(n)$
$\mathfrak{sp}(2n;\mathbb{C})$	$\mathfrak{usp}(2n)$	$\mathfrak{usp}(2p,2n-2p)$	$\mathfrak{usp}(2p)\oplus\mathfrak{usp}(2n-2p)$
$\mathfrak{sp}(2n;\mathbb{C})$	$\mathfrak{usp}(2n)$	$\mathfrak{sp}(2n)$	$\mathfrak{u}(n)$

2.4 Clebsch-Gordon

Consider the product $j=1$ with $j=\frac{1}{2}$ and calculate the Clebsch–Gordon coefficient (CGC) $<1\frac{1}{2},0\frac{1}{2}|\frac{1}{2},\frac{1}{2}>$ similar to the way in which we did it in the lecture (and book p.285) for the product of two $j=\frac{1}{2}$ representations. Derive from this the 3-j symbol where this CGC occurs. This $3j$-symbol is invariant under cyclic permutation of the 3 columns, and therefore the same one also can be derived from the CGC that we obtained in the lecture. Compare the results.

2.5 Young tableaux and product of representations

Consider the product of SU(4) representations



(5)

Calculate with Young tableaux, and obtain the product as a product of representations $n_1 \times n_2 = \ldots$ where n_1 and n_2 are the dimensions. Translate also in product in terms of representations defined by Dynkin labels, and obtain the product in that form. Which general rules can you recognize?