# RELATIVITY EXAM
# 29 January 2018

**<u>Oral Part (20 points):</u>** Discuss the Schwarzschild solution of the vacuum Einstein equations in 4 space-time dimensions and explain some of its important properties. Argue why it is of physical significance and theoretical importance. Is this solution unique? What are its symmetries?

## <u>Problem 1: The Weyl Tensor (8 points)</u>

The Weyl tensor in d space-time dimension is as:

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{d-2}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(d-1)(d-2)} g_{\rho[\mu}g_{\nu]\sigma}R$$

Take the Weyl tensor with one raised index, i.e. $C^{\rho}_{\sigma\mu\nu}$ and show that it is invariant under conformal transformations of the metric. This means that $C^{\rho}_{\sigma\mu\nu}$ for a general metric $g_{\mu\nu}$ is the same as the one for the metric $w^2(x)g_{\mu\nu}$. Here the scale factor w(x) is a general smooth non-vanishing function of the coordinates of the space-time.

Compute the Weyl tensor for flat Minkowski space $R^{1,\,d-1}$ and for d-dimensional anti de Sitter space $AdS_d$ and compare them. Can you explain your results?

(Use the fact that $AdS_d$ is a maximally symmetric space, which implies that its Riemann tensor obeys the identity:

$$R_{\rho\sigma\mu\nu} = \frac{R}{d(d-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)$$

and the Ricci scalar R is a constant.

## <u>Problem 2: Light Trajectories and Kerr (12 points)</u>

The Kerr solution of GR in vacuum is given by the line element

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{4GM\,a\,r\sin^2\theta}{\rho^2}dt d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

where $\Delta \equiv r^2 - 2GMr + a^2$ and $\rho^2 \equiv r^2 + a^2\cos^2\theta$.

Consider the orbits of massless particles with affine parameter $\lambda$, in the equatorial plane $(\theta = \pi/2)$ of a Kerr black hole.

1) Show that

$$\left(\frac{\mathrm{dr}}{\mathrm{d}\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4}(\mathrm{E} - \mathrm{LW} + (\mathrm{r}))(\mathrm{E} - \mathrm{LW} - (\mathrm{r}))$$

where $\Sigma^2 \equiv (\mathrm{r}^2 + \mathrm{a}^2)^2 - \mathrm{a}^2\Delta\sin^2\theta$. The constants E and L are the conserved energy and angular momentum of the massless particles. You have to find an explicit expression for the functions $W \pm (r)$.

Hint: Use that $dt$ and $d\phi$ are Killing vectors of the Kerr solutions. Use also that for a Killing vector K, the quantity $K_\mu \frac{dx^\mu}{d\lambda}$ is conserved along a geodesic.

2) Using this result and assuming that $\Sigma^2 > 0$ everywhere, show that the orbit of a photon in the equatorial plane cannot have a turning point inside the outer event horizon r+ .This implies that ingoing light rays cannot escape once they cross r+ and therefore this surface is really an event horizon.