Exam Statistical Mechanics
19 November 2019, 2-4pm

The total score is 20 points!

3 points

Diffusion from two sources

Consider a one-dimensional system with N diffusing particles. At time $t = 0$ we release $3N/4$ particles from $x = -a$ and $N/4$ from $x = a/2$ as illustrated in Fig. 1. Calculate τ, the time at which the diffusion current vanishes in $x = 0$.

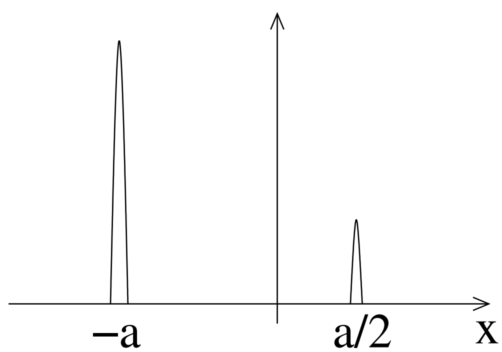

Figure 1:

2 points

Gas in gravitational field

Consider an ideal gas of N particles of mass m in a gravitational field with potential $\phi(z) = mgz$. The particles are in a cylindrical container of height L and radius R. The axis of the cylinder is oriented along the direction of the field ($0 \leq z \leq L$) Calculate the average density of particles

$$\rho(\vec{r}) = \langle \sum_{i=1}^{N} \delta(\vec{r} - q_i) \rangle$$

Consider two heights $z_1 \gg z_2$. Which of the following statements is true (justify!):

a) The average velocity of particles in z_1 is much higher than that of particles in z_2.

b) The average velocity of particles in z_2 is much higher than that of particles in z_1.

c) Particles will have the same average velocity independent on the height.

3 points

Kinetic energy of classical particle

We consider a system with N particles of mass m, in a container with volume V and at a temperature T.

a) Find the probability distribution $p(E_k)$ of the kinetic energy of a single particle $E_k = mv^2/2$.

b) Obtain from this the most probable value E_k^* and the average $\langle E_k \rangle$

4 points

Anharmonic oscillator

Although the harmonic oscillator is a widely used model in physics, it represents a quite idealized situation. It is applicable to systems which are weakly displaced from the minmal energy state. In more generality higher order terms may become relevant. We consider here a single particle in one dimension in an anharmonic potential:

$$\phi(x) = \frac{K}{2}x^2 + \frac{\Gamma}{4}x^4$$

Unfortunately, this model cannot be solved exactly. If the effect of Γ is weak, one can use perturbation theory. This consists in expanding the factor $\exp(-\beta\Gamma x^4/4)$ entering in the canonical partition function Z in Taylor series to lowest order in Γ. Using this approach

a) Calculate Z to first order Γ and obtain the average value of the energy $\langle E \rangle$ from it.

b) Calculate $\langle x^2 \rangle$ and $\langle x^4 \rangle$ to first order in Γ from the partition function Z obtained in a).

c) Using the results obtained in b) verify that the following quantity $K\langle x^2 \rangle + \Gamma\langle x^4 \rangle$ agrees with the equipartition theorem.

4 points

Gas in a two dimensional disk

We consider an ideal gas of N particles in a cylindrically shaped container of radius R and height h. We assume that h is very small so that the gas can be effectively considered as confined in a two dimensional disk. The particles are also subject to an external harmonic force which attracts them towards the center of the disk (placed at the origin of the coordinates). The Hamiltonian for one particle is

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + k\frac{{q_x}^2 + {q_y}^2}{2}$$

where k is the effective spring constant of the force and m is the mass of the particles.

a) Calculate the internal energy, pressure, and specific heat.

b) Calculate the probability that a given particle is contained in an annulus of radii $a < r < b$.

4 points

Virial coefficients for Hard Rods

In the Hard Rod model two particles with positions x_1 and x_2 interact through the potential $\Phi(x_1, x_2) = \varphi(|x_1 - x_2|)$ with:

$$\varphi(r) = \begin{cases} +\infty & \text{for} \quad r < \sigma \\ 0 & \text{for} \quad r > \sigma \end{cases}$$

We consider a system of N rods at a temperature T in a line of length L. The configurational partition function for this system can be computed exactly and it is equal to

$$Q(N, L, T) = (L - N\sigma)^N$$

a) Calculate the total (=kinetic + potential) energy of the system.

b) Calculate the equation of state of the system $P = P(N, L, T)$ expressing the pressure as a function of N, L and T.

c) Calculate the second and third virial coefficients.

d) Show that the coefficients in c) can also be obtained from an appropriate limiting behavior of the exact equation of state.

Hint: The 2nd and 3rd virial coefficients in one dimension are defined as follows

$$b_2 = -\frac{1}{2}\int_{-\infty}^{+\infty} dx\, f(|x|) \qquad b_3 = -\frac{1}{3}\int\int_{-\infty}^{+\infty} dx dx' f(|x|) f(|x'|) f(|x - x'|)$$

where $f(r) \equiv \exp(-\beta\varphi(r)) - 1$.