Juni 2021 Operator Algebras

Information

The best 3 questions are considered for the final mark.
The official notes and unsolved excercise sheets can be used.

1

Let A be a $C^\star$ algebra. Prove the equivalence between:

- $A \neq \mathbb{C}1$
- There exist positive a, b of norm 1 so that $ab = 0$
- There exists a unitary u with $\{-1, 1\} \subset \sigma(u)$

2

Let A be a $C^\star$ algebra.

- Show that every character on A is a pure state.
  Hint: $\mathbb{T}$ are the extreme points of the closed unit disk.
- If A is noncommutative, show that there exists a pure state that is not a character.

3

Let $A \subset \mathcal{L}(\mathcal{H})$ a concrete $C^\star$ algebra with $1 \in A$. Let $M = A''$. Prove that for every unitary $u \in M$, there is a net v_λ in A converging to u in the Strong Operator Topology.
Hint: $exp(it) = cos(t) + isin(t)$ for real t

4

Let $\varepsilon \in [0, 1)$.

- The universal $A_\varepsilon = C^\star(s_\varepsilon \mid ||s_\varepsilon^\star s_\varepsilon - 1|| < \varepsilon)$ exists
- $\forall \delta \in (\varepsilon, 1)$, there exists a surjective $\star$homomorphism $A_\delta \to A_\varepsilon$ with $s_\delta \to s_\varepsilon$
- If $\varepsilon_n \to 0$ is any decreasing sequence converging to zero, the limit of the inductive system $\{A_{_n} \to A_{\varepsilon_{n+1}}\}$ is isomorphic to the Toepliz algebra $\mathcal{T}$