QUANTUM MECHANICS — EXAM

1. Bosonic and fermionic harmonic oscillators: In class you saw that the bosonic SHO has Hamiltonian

$$H_B = \frac{\hbar\omega}{2}(a^\dagger a + a a^\dagger), \qquad [a, a^\dagger] = 1,\ [a, a] = 0,\ [a^\dagger, a^\dagger] = 0 \tag{.1}$$

and gave an interpretation of the operator $N_B = a^\dagger a$ that its eigenvalues reads out the "level of excitement" $N_B |n\rangle = n |n\rangle$. In quantum field theory N_B is called the number operator and its eigenvalues are instead interpreted to count the number of particles n. Consider now the fermionic SHO Hamiltonian of the form

$$H_F = \frac{\hbar\omega}{2}(b^\dagger b - b b^\dagger), \qquad \{b, b^\dagger\} = 1,\ \{b, b\} = 0,\ \{b^\dagger, b^\dagger\} = 0, \tag{.2}$$

where curly brackets denote the anti-commutator $\{A, B\} = AB + BA$.

(a) Show that the fermionic number operator $N_F = b^\dagger b$ satisfies

$$N_F(N_F - 1) = 0 \tag{.3}$$

[Hint: first show that $N_F^2 = N_F$]. Use this result to argue that the eigenvalues of N_F are 0 and 1.

(b) Denoting the respective eigenstates $|0\rangle$ and $|1\rangle$ (such that $N_F |0\rangle = 0 |0\rangle$ and $N_F |1\rangle = 1 |1\rangle$) show that

$$b^\dagger |0\rangle = |1\rangle, \qquad b |1\rangle = |0\rangle, \qquad b^\dagger |1\rangle = 0, \qquad b |0\rangle = 0. \tag{.4}$$

(c) Using the results from 1b show that

$$H_F |0\rangle = -\frac{\hbar\omega}{2} |0\rangle, \qquad H_F |1\rangle = \frac{\hbar\omega}{2} |1\rangle. \tag{.5}$$

Consider now a bosonic system and a fermionic system of infinitely many, non-interacting, oscillators whose Hamiltonians read

$$\begin{aligned} H_B &= \sum_{j=1}^{\infty} \frac{\hbar\omega}{2}(a_j^\dagger a_j + a_j a_j^\dagger), \qquad [a_i, a_j^\dagger] = \delta_{ij},\ [a_i, a_j] = 0,\ [a_i^\dagger, a_j^\dagger] = 0, \\ H_F &= \sum_{j=1}^{\infty} \frac{\hbar\omega}{2}(b_j^\dagger b_j - b_j b_j^\dagger), \qquad \{b_i, b_j^\dagger\} = \delta_{ij},\ \{b_i, b_j\} = 0,\ \{b_i^\dagger, b_j^\dagger\} = 0 \end{aligned} \tag{.6}$$

in the respective cases. The eigenstates of these systems are labelled as $|n_1, n_2, n_3, \ldots\rangle$ and the vacuum state is labelled as $|\text{vac}\rangle = |0, 0, 0, \ldots\rangle$ such that, for example,

$$a_2^\dagger |0, 0, 0, \ldots\rangle = |0, 1, 0, \ldots\rangle. \tag{.7}$$

(d) Show that the relations

$$\begin{aligned} a_i^\dagger a_j^\dagger |\text{vac}\rangle &= a_j^\dagger a_i^\dagger |\text{vac}\rangle \\ b_i^\dagger b_j^\dagger |\text{vac}\rangle &= -b_j^\dagger b_i^\dagger |\text{vac}\rangle \end{aligned} \tag{.8}$$

are consistent with the commutator (anti-commutator) appearing in the algebra of the bosonic (fermionic) ladder operators. Argue that $n_i \in \{0, 1, 2, \ldots\}$ in the bosonic case and $n_i \in \{0, 1\}$ in the fermionic case [Hint: consider $i = j$ in equation (.8)].

Solution:

(a) For this part we simply use the anti-commutation relations that b and $b^\dagger$ satisfy

$$\begin{aligned} N_F^2 &= b^\dagger b b^\dagger b \\ &= b^\dagger b(1 - bb^\dagger) \\ &= b^\dagger b - b^\dagger b b b^\dagger \\ &= b^\dagger b = N_F, \end{aligned} \tag{.9}$$

where crucially in the last line we have used the fact the $\{b, b\} = 2bb = 0$. Thus, we indeed have equation (.3). Now the trick is to use an eigenbasis of N_F, for which $N_F |n\rangle = n |n\rangle$. Applying (.3) to an eigenstate we get

$$N_F(N_F - 1) |n\rangle = n(n-1) |n\rangle \equiv 0, \tag{.10}$$

thus the eigenvalues of N_F are indeed 1 or 0.

(b) First we recognise that

$$\langle 0|N_F|0\rangle = \langle 0|b^\dagger b|0\rangle = (\langle 0| b^\dagger)(b|0\rangle) = 0 \qquad \Longrightarrow b|0\rangle = 0 \tag{.11}$$
$$\langle 1|N_F|1\rangle = \langle 1|1 - bb^\dagger|1\rangle = 1 - (\langle 1| b)(b^\dagger |1\rangle) = 1 \qquad \Longrightarrow b^\dagger |1\rangle = 0. \tag{.12}$$

Then we postulate that

$$b^\dagger |0\rangle = \alpha |0\rangle + \beta |1\rangle \tag{.13}$$
$$b|1\rangle = \gamma |0\rangle + \delta |1\rangle \tag{.14}$$

and we write

$$0 = b^\dagger b |0\rangle = |0\rangle - bb^\dagger |0\rangle = |0\rangle - b(\alpha |0\rangle + \beta |1\rangle) = |0\rangle - \beta\gamma |0\rangle - \beta\delta |1\rangle \qquad \Longrightarrow \delta = 0, \quad \beta\gamma = 1 \tag{.15}$$
$$|1\rangle = b^\dagger b |1\rangle = b^\dagger \gamma |0\rangle = \gamma\alpha |0\rangle + \gamma\beta |1\rangle \qquad \Longrightarrow \alpha = 0, \quad \beta\gamma = 1. \tag{.16}$$

Finally, we assume that the states $|0\rangle$ and $|1\rangle$ are properly normalized: $\langle 0|0\rangle = 1$ and $\langle 1|1\rangle = 1$, and we demand that $\gamma |0\rangle$ and $\beta |1\rangle$ are also normalized. This imposes

$$\langle 1|b^\dagger b|1\rangle = \langle 0|0\rangle |\gamma|^2 = 1 \qquad \Longrightarrow \gamma^*\gamma = 1 \tag{.17}$$
$$\langle 0|bb^\dagger|0\rangle = \langle 1|1\rangle |\beta|^2 = 1 \qquad \Longrightarrow \beta^*\beta = 1. \tag{.18}$$

We finally conclude that

$$\beta = e^{i\theta}, \qquad \gamma = e^{-i\theta} \tag{.19}$$

with real θ is a solution to the above equations. The canonical choice in the literature is $\theta = 0$.

(c) Now this part is easy

$$\frac{1}{2}\hbar\omega(b^\dagger b - bb^\dagger) |0\rangle = \hbar\omega\left(N_F - \frac{1}{2}\right) |0\rangle = -\frac{\hbar\omega}{2} |0\rangle \tag{.20}$$
$$\frac{1}{2}\hbar\omega(b^\dagger b - bb^\dagger) |1\rangle = \hbar\omega\left(N_F - \frac{1}{2}\right) |1\rangle = +\frac{\hbar\omega}{2} |1\rangle \tag{.21}$$

(d) (.8) is immediately obviously consistent when we write

$$\begin{aligned} 0 &= a_i^\dagger a_j^\dagger |\text{vac}\rangle - a_j^\dagger a_i^\dagger |\text{vac}\rangle = [a_i^\dagger, a_j^\dagger] |\text{vac}\rangle \\ 0 &= b_i^\dagger b_j^\dagger |\text{vac}\rangle + b_j^\dagger b_i^\dagger |\text{vac}\rangle = \{b_i^\dagger, b_j^\dagger\} |\text{vac}\rangle. \end{aligned} \tag{.22}$$

For each independent SHO there is no restrictions for how may times we can raise with $a_j^\dagger$ from $|0\rangle$, thus $n_j = \{0, 1, 2, \dots\}$ in the bosonic case, while we can obly raise once with $b_j^\dagger$, thus $n_j = \{0, 1\}$ in the fermionic case.

2. Landau levels in the Landau gauge: The Hamiltonian of a free electron in a constant magentic field $\mathbf{B}$ is given by

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2, \qquad \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}. \tag{.23}$$

In this problem we will choose $\mathbf{B} = B\hat{\mathbf{z}}$, thus the electron is restricted to moving in the $x - y$ plane. Further we work in the Landau gauge specified by

$$\mathbf{A} = xB\hat{\mathbf{y}} \tag{.24}$$

and ignore the effects of spin. Throughout his problem you might need to use that the eigenunctions of the SHO with Hamiltonian $H_{\text{SHO}} = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2$ are given by (up to normalization)

$$\phi_n(x) \sim e^{-\frac{m\omega}{2\hbar}x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \qquad n = 0, 1, 2, \dots \tag{.25}$$

(a) Explicitly writing down the Hamiltonian in terms of $p_x = -i\hbar\partial_x$ and $p_y = -i\hbar\partial_y$ and taking the eigenfunction ansatz

$$\Psi_{n,k}(x, y) = e^{iky}\psi_n(x) \tag{.26}$$

show that the Hamiltonian is simply the one of a shifted Harmonic oscillator

$$H = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_B^2(x + kl_B^2)^2. \tag{.27}$$

In this process spell out explicit formulas for the constants ω_B and l_B and give their physical intepretation.

(b) Spell out explicitly the eigenfunctions $\psi_n(x)$ in terms of the eigenfunctions of the unshifted SHO $\phi_n(x)$. Spell out their corresponding eigenenergies. What are the allowed values of n and k? Are the energy levels degenerate?

(c) Applying an additional electric field in the x direction $\mathbf{E} = E\hat{\mathbf{x}}$ introduces a term in the Hamiltonian of the form eEx. Complete the square to argue that the new wavfunction are given by

$$F_{n,k}(x, y) = \Psi_{n,k}(x + \Delta x, y) \tag{.28}$$

and give explicit formula fo Δx. Spell out the new energy levels of the system. Is there any degeneracy left?

Solution:

(a) The electron moves only the $x - y$ plane thus the Hamiltonian is given by

$$H = \frac{1}{2m}\left[(p_x + eA_x)^2 + (p_y + eA_y)^2\right] = \frac{1}{2m}\left[p_x^2 + (p_y + eBx)^2\right]. \tag{.29}$$

In the position basis this operator looks like

$$H = \frac{1}{2m}\left[-\hbar^2\partial_x^2 + (-i\hbar\partial_y + eBx)^2\right]. \tag{.30}$$

Acting on the ansatz wavefunction we get

$$\frac{1}{2m}\left[-\hbar^2\partial_x^2 + (-i\hbar\partial_y + eBx)^2\right]e^{iky}\psi_n(x) = \frac{1}{2m}\left[-\hbar^2\partial_x^2 + (\hbar k + eBx)^2\right]e^{iky}\psi_n(x). \tag{.31}$$

Thus the Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}\frac{e^2B^2}{m}\left(k\frac{\hbar}{eB} + x\right)^2 \tag{.32}$$

and we identify

$$\begin{aligned} \omega_B &= \frac{eB}{m}, \qquad \text{the magnetron frequency} \\ l_B &= \sqrt{\frac{\hbar}{eB}}, \qquad \text{a lengthscale of quantum phenomena in a magnetic field.} \end{aligned} \tag{.33}$$

(b) Schrödinger's equation reads

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_B^2\left(x + kl_B^2\right)^2\right]\psi_n(x) = E\psi_n(x). \tag{.34}$$

Since the potential is just shifted in the x direction, the wavefunctions of the shifted potential are just the shifted wavefunctions of the unshifted potential, thus

$$\psi_n(x) = \phi_n(x + kl_B^2) \sim e^{-\frac{m\omega}{2\hbar}(x+kl_B^2)^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x + kl_B)^2\right). \tag{.35}$$

The energy only depends on the quantum number $n = 0, 1, 2, \ldots$, while the quantum number $k \in \mathbb{R}$, since the waves in y direction e^{iky} are allowed to extend infinitelly. Thus there is a huge degeneracy coming from all values of k for each n. The eigenergies are those of the SHO

$$E = E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{.36}$$

(c) With the electric field applied the Hamiltonian reads

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_B^2\left(x + kl_B^2\right)^2 + eEx \\ &= -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_B^2\left(x + kl_B^2 + \frac{eE}{m\omega_B^2}\right)^2 - eE\left(kl_B^2 + \frac{eE}{2m\omega_B^2}\right) \\ &= -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_B^2\left(x + kl_B^2 + \frac{mE}{eB^2}\right)^2 - eE\left(kl_B^2 + \frac{mE}{2eB^2}\right), \end{aligned} \tag{.37}$$

where in the second equality we completed the square and in the final equality we strategically substituted the expresion for ω_B. We can identify

$$\Delta x = \frac{mE}{eB^2}. \tag{.38}$$

Schrödinger's equation reads

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega_B^2\left(x + kl_B^2 + \Delta x\right)^2 - e\mathcal{E}\left(kl_B^2 + \Delta x/2\right)\right]F_{n,k}(x) = EF_{n,k}(x), \tag{.39}$$

thus the wavefunctions are given by

$$F_{n,k}(x) = e^{iky}\phi_n(x + kl_B^2 + \Delta x) \tag{.40}$$

and the eigenenergies are now non-degenerate

$$E_{n,k} = \hbar\omega_B\left(n+\frac{1}{2}\right) - e\mathcal{E}(kl_B^2 + \Delta x/2) \tag{.41}$$

3. Failure of perturbation theory, triumph for WKB: Consider a double well potential of the form

$$V(x) = \lambda^2(x^2-a^2)^2. \tag{.42}$$

We want to find the energy eigenstates of a particle with $E < V(0)$. We will focus on the limit of large a and large λ in which case the system effectively behaves as two independent SHOs with small coupling between them due the tunneling through the barrier.

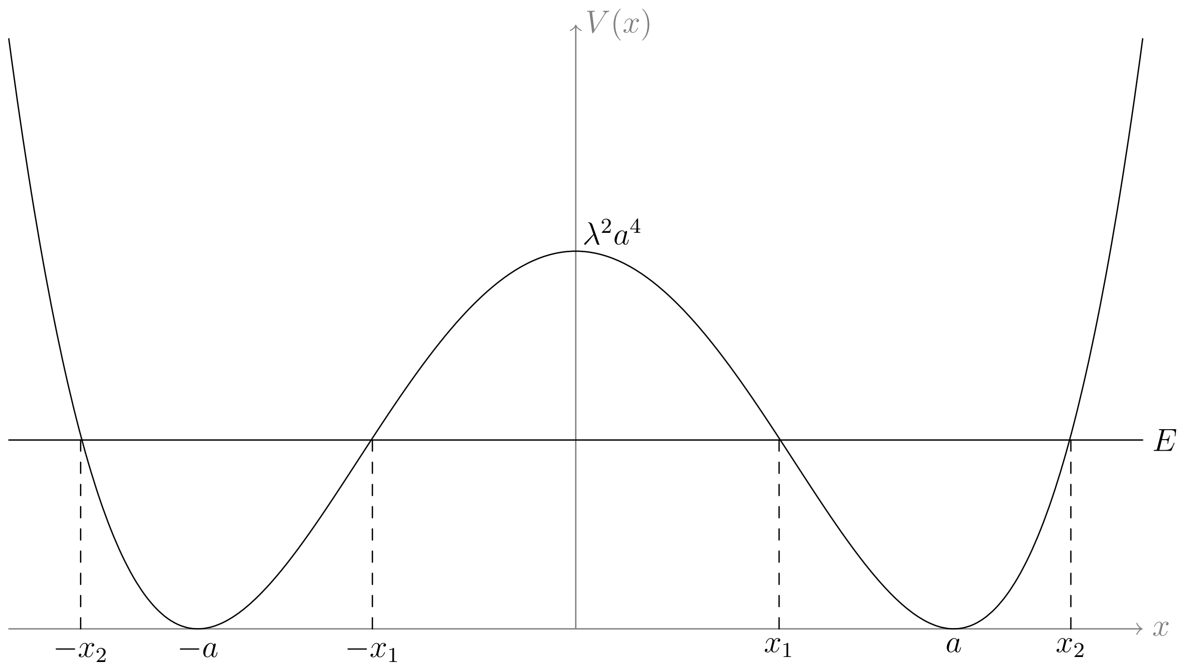

FIG. 1. Coupling to flat

(a) Say that we initially neglect tunneling. Based on the symmmetry of the potential argue that you expect a symmetric and an antisymmetric wavefunction that are degenerate for a given energy E. When tunneling is reintroduced the degeneracy between these energy levels is lifted. Based on your physical intuition which one of the two do you expect to have higher energy?

(b) Argue briefly why perturbation theory around any of the two minima fails to give the correct energy levels.

(c) Using the WKB connection formulae one can derive that the wavefunction is

$$\phi(x) = \begin{cases} \dfrac{1}{\sqrt{p(x)}}\exp\left(-\dfrac{1}{\hbar}\displaystyle\int_{x_2}^{x}\mathrm{d}z\,|p(z)|\right), & x > x_2 \\ \dfrac{2}{\sqrt{p(x)}}\sin\left(-\dfrac{1}{\hbar}\displaystyle\int_{x}^{x_2}\mathrm{d}z\,p(z) + \dfrac{\pi}{4}\right), & x_1 < x < x_2 \\ \dfrac{1}{\sqrt{p(x)}}\left[2\cos\theta\exp\left(\dfrac{1}{\hbar}\displaystyle\int_{x}^{x_1}\mathrm{d}z\,|p(z)|\right) + \sin\theta\exp\left(-\dfrac{1}{\hbar}\displaystyle\int_{x}^{x_1}\mathrm{d}z\,|p(z)|\right)\right], & x < x_1, \end{cases} \tag{.43}$$

where

$$p(x) = \sqrt{2m(E - V(x))}, \qquad \theta = \frac{1}{\hbar}\int_{x_1}^{x_2} \mathrm{d}z\, p(z). \tag{.44}$$

Since we expect symmetric and antisymmetric wavefunctions ($\phi_+(x)$ and $\phi_-(x)$ respectively) show that the conditions

$$\phi_+'(0) = 0, \qquad \phi_-(0) = 0 \tag{.45}$$

demand

$$\tan\theta = \pm 2e^{\gamma}, \qquad \gamma = \frac{1}{\hbar}\int_{-x_1}^{x_1} \mathrm{d}z\, |p(z)|, \tag{.46}$$

where "+" is for the symmetric solution and "−" for the antisymmetric. You might need to use that $|p(x)|$ is symmetric and

$$\left.\frac{\mathrm{d}|p(x)|}{\mathrm{d}x}\right|_{x=0} = 0. \tag{.47}$$

(d) If we had only one SHO WKB teaches us that

$$\theta = \left(n + \frac{1}{2}\right)\pi + \epsilon, \tag{.48}$$

with $\epsilon = 0$. When we allow tunneling this condition is modified and we parametize the modification via the small number ϵ. Argue that for a high and wide barrier

$$\tan\theta = -\frac{1}{\epsilon}. \tag{.49}$$

You might need the relations

$$\sin\left(\left(n + \frac{1}{2}\right)\pi + \epsilon\right) = (-1)^n \cos(\epsilon), \qquad \cos\left(\left(n + \frac{1}{2}\right)\pi + \epsilon\right) = (-1)^{n+1}\sin(\epsilon). \tag{.50}$$

Argue that this implies that

$$\theta = \left(n + \frac{1}{2}\right)\pi \mp \frac{1}{2}e^{-\gamma}. \tag{.51}$$

NB: "−" above is for the symmetric solution and "+" for the antisymmetric!

(e) To evaluate θ approximate the potential on the right side ($x > 0$) as

$$V(x) \approx \frac{1}{2}m\omega^2(x - a)^2 \tag{.52}$$

and perform the θ integral. You might need the folrmula

$$\int_0^{z_*} \mathrm{d}z\, \sqrt{z_*^2 - z^2} = \frac{\pi}{4}z_*^2. \tag{.53}$$

As such spell out the energies in terms of n and γ.

(f) Suppose that the particle initially is locallized in the right well in the state

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}(\phi_+(x) + \phi_-(x)). \tag{.54}$$

After how long you will see that particle transitioned to the left well in state

$$\Psi(x, t = \Delta t) = \frac{1}{\sqrt{2}}(\phi_+(x) - \phi_-(x))? \tag{.55}$$

Give physical interpretation of γ.

Solution:

(a) Potential is symmetric $\implies$ wavefunctions should be symmetric or antisymmetric by theory of differential equations. When tunneling is reintroduced the antisymmmetric solution should have higher energy, since it will have one more "node" compared to the symmetric solution whose energy is closest to it.

(b) Perturbation theory about each minima sees only the energy levels of one SHO, they are not degenerate. We can write the potenatial as SHO + correction and calcualte the energy shift, but this will no capture the degenerecay between symmetric and antisymmetric solutions of the full potential. Perturbation theory is doomed.

(c) Solving for the antysymmetric condition is easier, so we start there. One gets

$$2\cos\theta \exp\left(\frac{1}{\hbar}\int_0^{x_1} \mathrm{d}z\, |p(z)|\right) + \sin\theta \exp\left(-\frac{1}{\hbar}\int_0^{x_1} \mathrm{d}z\, |p(z)|\right) = 0, \tag{.56}$$

which can be rearanged as

$$\tan\theta = -2\exp\left(\frac{2}{\hbar}\int_0^{x_1} \mathrm{d}z\, |p(z)|\right) = -2e^{\gamma}, \tag{.57}$$

where in the last equality we have used the fact that $p(x)$ is symmetric around $x = 0$. For the symmetric solution one gets

$$\begin{aligned} 0 = &\left\{-\frac{1}{2|p(x)|^{3/2}}\frac{\mathrm{d}|p(x)|}{\mathrm{d}x}\left[2\cos\theta \exp\left(\frac{1}{\hbar}\int_x^{x_1} \mathrm{d}z\, |p(z)|\right) + \sin\theta \exp\left(-\frac{1}{\hbar}\int_x^{x_1} \mathrm{d}z\, |p(z)|\right)\right]\right\}_{x=0} \\ &+ \left\{\frac{1}{|p(x)|^{1/2}}\left[2\cos\theta \exp\left(\frac{1}{\hbar}\int_x^{x_1} \mathrm{d}z\, |p(z)|\right)\left(-\frac{|p(x)|}{\hbar}\right) + \sin\theta \exp\left(-\frac{1}{\hbar}\int_x^{x_1} \mathrm{d}z\, |p(z)|\right)\left(\frac{|p(x)|}{\hbar}\right)\right]\right\}_{x=0}. \end{aligned} \tag{.58}$$

Using the facts that $\mathrm{d}|p(0)|\,/\,\mathrm{d}x = 0$ we get simply

$$-2\cos\theta \exp\left(\frac{1}{\hbar}\int_0^{x_1} \mathrm{d}z\, |p(z)|\right) + \sin\theta \exp\left(-\frac{1}{\hbar}\int_0^{x_1} \mathrm{d}z\, |p(z)|\right) = 0, \tag{.59}$$

which can again be rearanged to

$$\tan\theta = 2e^{\gamma}. \tag{.60}$$

(d) Using (.50) this is rather easy

$$\tan\theta = \frac{\sin\left((n+\frac{1}{2})\pi + \epsilon\right)}{\cos\left((n+\frac{1}{2})\pi + \epsilon\right)} = \frac{(-1)^n\cos(\epsilon)}{(-1)^{n+1}\sin(\epsilon)} \approx -\frac{1}{\epsilon}, \tag{.61}$$

since for small ϵ we have $\cos(\epsilon) \approx 1$ and $\sin(\epsilon) \approx \epsilon$. Since $\tan\theta = \pm 2e^{\gamma}$ we indeed obtain $\epsilon = \mp\frac{1}{2}e^{-\gamma}$.

(e) With our approximate potential on the right side we get that the two turning points x_1 and x_2 are approximated by

$$E = \frac{1}{2}m\omega^2(x-a)^2 \quad \implies \quad x_\pm = a \pm \sqrt{2E/m\omega^2}. \tag{.62}$$

Thus the θ integral is approximated to be

$$
\begin{aligned}
\theta &\approx \frac{1}{\hbar}\int_{x_-}^{x_+} \mathrm{d}z\,\sqrt{2mE - m^2\omega^2(x-a)^2} \\
&= \frac{1}{\hbar}\int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}} \mathrm{d}u\,\sqrt{2mE - m^2\omega^2 u^2} \\
&= \frac{1}{m\omega\hbar}\int_{-\sqrt{2Em}}^{\sqrt{2Em}} \mathrm{d}v\,\sqrt{2mE - v^2} \\
&= \frac{2}{m\omega\hbar}\int_{0}^{\sqrt{2Em}} \mathrm{d}v\,\sqrt{2mE - v^2} \\
&= \frac{2}{m\omega\hbar}\frac{\pi}{4}2mE = \frac{\pi E}{m\omega}.
\end{aligned}
$$

Thus, we finally obtain

$$
E_\pm = \left(n + \frac{1}{2}\right)\hbar\omega \mp \frac{\hbar\omega}{2\pi}e^{-\gamma} \tag{.63}
$$

and confirm that the symmetric solution (the "+") has smaller energy.

(f) To get the time evolution we write

$$
\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2}}\left(\phi_+ e^{-iE_+t/\hbar} + \phi_- e^{-iE_-t/\hbar}\right) \\
&= \frac{e^{-iE_+t/\hbar}}{\sqrt{2}}\left(\phi_+ e^{-iE_+t/\hbar} + \phi_- e^{-i(E_- - E_+)t/\hbar}\right).
\end{aligned} \tag{.64}
$$

The first time we get a minus in front of the second term is when $t = \Delta t$ such that

$$
e^{-i(E_- - E_+)\Delta t/\hbar} = e^{-i\pi}, \tag{.65}
$$

or equivalently

$$
\Delta t = \frac{\pi\hbar}{E_- - E_+} = \frac{\pi^2}{\omega}e^{\gamma}. \tag{.66}
$$

We see that γ can be interpreted as the log of the tunneling time. It also happens to be the classical action, but that's another story.