Relativity Exam

28 January 2019

1 First part oral

The Schwarzschild metric describes a solution of the Einstein Equation in four-dimensional space-time. Describe the symmetries of this metric and the defining properties as well as the physical relevance of this solution. Indicate whether this solution is somehow unique and explain.

2 Second part written

2.1 Problem 1

Given manifold described by the product of a two-dimensional Anti-de-Sitter space and two-dimensional Sphere written formally as: AdS $\times \mathcal{S}^2$ and equipped with the following metric:

$$\mathrm{d}s^2 = \frac{-\,\mathrm{d}t^2 + \mathrm{d}y^2}{y^2} + \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2$$

Find all the Killing vectors explicitly and explain how many Killing vectors there are in total.

2.2 Problem 2

Given the Kerr metric of a rotating black-hole:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho}\right) dt^2 - \frac{4GMra\sin^2\theta}{\rho}\, dt\, d\phi + \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2GMra^2}{\rho}\sin^2\theta\right)\sin^2\theta\ d\phi^2 \end{aligned}$$

where $\rho = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2GMr + a^2$ in this context. We consider a massless particle in the equatorial plane, where $\theta = \frac{\pi}{2}$ and r greater than zero.

1. Find r_+ and r_- the event horizons for this metric.
2. Given conserved values E and L for a massless particle on geodesic with affine parameter λ prove the following equation.

   $$\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4}(E - W_-(r)L)(E - W_+(r)L)$$

   Solve for the functions $W_-(r)$ and $W_+(r)$ explicitly, given that $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2(\theta)$ (or something as such this might be incorrect).
3. Consider a photon with a certain energy E and a certain angular momentum L in orbit around the black-hole, show that if the photon is between r_+ and r_- that it is not possible for it to turn around in this region and hence r_+ a real event horizon.