Symmetries in Quantum Mechanics

Final Exam — Friday January 17, 2014

- The exam is written and open book. You get three hours and thirty minutes.
- Please write your name on each page, write clearly and put your final answer in a box.
- Try to be brief — answers do not have to be exhaustively complete to be sufficiently complete.

1. Answer yes or no and give a brief explanation:

   (a) A particle moves in a spherically symmetric potential. If the x-component of the angular momentum is positive with 100% certainty at time $t = 0$, can it be found to be negative at a later time?

   (b) The Hamiltonian of a free particle has rotation and translation invariance. Is it possible to simultaneously diagonalize the energy, the z-component of the angular momentum and the x-component of the momentum?

   (c) A lattice consists of $10^8 \times 10^8 \times 10^8$ spin 1/2 degrees of freedom at fixed positions. Is it possible for the system to have total spin zero (i.e. $\mathbf{S}^2 = 0$)?

   (d) A hydrogen atom is placed in a weak constant electric field. Will there be any degeneracy left of the originally 4-fold degenerate $n = 2$ energy level? (Ignore proton and electron spin degrees of freedom.)

   (e) Momentum is conserved for systems with continuous translation symmetry. Does there exist a conserved quantity for systems with discrete translation symmetry?

   (f) Let ψ be some state of the hydrogen atom (not necessarily an energy eigenstate), and $\mathbf{r}$ the relative position of proton and electron. Can $\langle\psi|\mathbf{r}|\psi\rangle$ ever be nonzero?

2. What is the minimum value of the total angular momentum $\mathbf{J}^2$ that a state must have in order to have a nonzero expectation value for an operator $\mathcal{O}_j^m$ of spin j?

3. Consider a chain of 10 spin 1/2 particles at fixed position. Give an example of a state of total spin 5 and a state of total spin 4.

4. A hydrogen atom is placed in a weak electric field with potential $V(x, y, z) = \epsilon xyz$, with ϵ a constant. *(a)* What are the selection rules on matrix elements $\langle n\ell m|V|n'\ell' m'\rangle$? *(b)* Sketch how you would use Wigner-Eckart to compute such matrix elements.

5. Two identical spin 1/2 particles of mass m are bound together by a quadratic potential, with no direct interactions between the spin vectors $\mathbf{S_1}$, $\mathbf{S_2}$ of the two particles. The Hamiltonian is thus given by

$$H = \frac{1}{4m}\mathbf{P}^2 + \frac{1}{m}\mathbf{p}^2 + \frac{m\omega^2}{4}\mathbf{r}^2,$$

   where $\mathbf{P}$ denotes the center of mass momentum and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$ the relative position and momentum. *(a)* What are the symmetries of the system leading to conserved quantities? *(b)* Describe the ground state and the first excited energy level of this system, or more precisely give their energy, degeneracy, and total angular momentum. Take into account the fact that the particles are identical fermions.

6. (Optional.) Tell me something interesting.