1 Oral part

1.1 Grand Canonical Ensemble

Show how the average number of particles $\langle N \rangle$ and the variance of N

$$\sigma_N = \left\langle N^2 \right\rangle - \langle N \rangle^2$$

can be calculated from the grand canonical partition function Ξ.

Now consider a system of non-interacting particles and assume that the one-particle $Z_1(V,T)$ is known. Compute Ξ and show that the relative fluctuations of the number of particles is small.

1.2 Bosons

Discuss the formula

$$n\lambda_T^3 = \frac{\lambda_T^3}{V}\frac{z}{1-z} + \sum_{l=1}^{\infty}\frac{z^l}{l^{3/2}},$$

where $z = e^{\beta\mu}$, in the two limits:

- with low density and high temperature, i.e. n $\lambda_T^3 \gg 1$
- with high density and low temperature, i.e. $1 \gg n\lambda_T^3$.

Pay in particular attention to the ground state.

2 Written part

2.1 Diffusion on a Linear potential

Consider particles in a potential $V(x) = -ax$ with diffusion coefficient D. Show that

$$c(x,t) = \frac{N}{\sqrt{4Dt}} e^{-\frac{(x-vt)^2}{4Dt}},$$

is a solution of the drift-diffusion equation and find v. Plot this solution for different times. Show that $F = \gamma v$ where F is the force on the particles and γ is the friction coefficient.

2.2 One-dimensional Harmonic oscillators

Consider a single one-dimensional oscillator in the canonical ensemble and calculate the average energy $\langle E \rangle$ and the variance of E, $\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2$.

Do the same for N independent harmonic oscillators.

2.3 Triatomic molecules

Consider a system of N molecules in a volume and temperature V. The molecules each consist of three atoms with masses m_1, m_2 and m_3. The atoms interact through a potential

$$\Phi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{\kappa}{2}(\mathbf{x}_1 - \mathbf{x}_2)^2 + \frac{\Gamma}{2}(\mathbf{x}_2 - \mathbf{x}_3)^2$$

- Calculate the partition function of the canonical ensemble. Also calculate the internal energy of the system and the specific heat c_V.
- Calculate the averages $\left\langle (\mathbf{x}_1 - \mathbf{x}_2)^2 \right\rangle$, $\left\langle \mathbf{x}_2 - \mathbf{x}_3)^2 \right\rangle$ and $\mathbf{x}_3 - \mathbf{x}_1)^2$.
- Calculate the pressure of the system.

2.4 The specific heat at low temperatures

Consider a system of N particles. Now calculate the partition function, using the approximation the only the lowest to energy states ϵ_1 and $\epsilon_2 = \epsilon_1 + \delta\epsilon$ contribute. Calculate the average energy and the specific heat. What happens to the specific heat at low temperatures?

2.5 The sun and blackbody radiation

In the course we derived the following energy density per unit volume with a frequency ω for the spectrum of a black body,

$$\epsilon_\omega d\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega.$$

- Using this formula and the fact that $\omega\lambda = 2\pi c$, derive the energy density $\epsilon d\lambda$ with a wavelength λ.

- Use $\frac{d\epsilon_\omega}{d\omega} = 0$ and $\frac{d\epsilon_\lambda}{d\lambda} = 0$ to estimate a value for the frequency ω_{max} and λ_{max} where ϵ_ω and ϵ_λ, respectively, is maximum. You can use the following numerical solutions, to the equations

$$\begin{aligned} x - 3(1 - e^{-x}) &= 0 \qquad\qquad & x &= A \\ x - 5(1 - e^{-x}) &= 0 \qquad\qquad & x &= B \end{aligned}$$

[1]

- Estimate the temperature of the sun assuming that $\lambda_{\text{max}} = ...$ [2]

[1] Op het examen waren uiteraard de precieze waarden A en B gegeven, maar dat doet er niet zoveel toe.

[2] Alle constanten die je nodig had voor deze vraag waren gegeven.