TEST Adv QM, 4 November 2020

PLEASE AVOID USING YOUR NOTES OR BOOK!

1 Approximate (3pts)

Consider the Hamiltonian (for $x \in \mathbb{R}$)

$$H_b = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right) + bx^4$$

Approximate its ground-state energy E_b in a reasonable way for $b > 0$. You could use a variational approach or anything else you judge appropriate.

2 Qubit (4pts)

a) Consider a single qubit. Show that any unitary operator U on $\mathbb{C}^2$ can be written as a rotation in spin space, up to a constant phase factor. That is, there exist $\alpha, \theta \in \mathbb{R}$ and a unit vector $\mathbf{n}$ such that:

$$U = \exp(\mathrm{i}\alpha)\exp(\mathrm{i}\theta\mathbf{n}\cdot\boldsymbol{\sigma}) \tag{1}$$

b) Calculate $\mathrm{Tr}[\,U\,\sigma_x]$.

Here are the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3 Density matrix (4pts)

Consider the state

$$|\psi\rangle = a|01\rangle + b|10\rangle \tag{2}$$

with $|a|^2 + |b|^2 = 1$.

- For which values of a and b is $|\psi\rangle$ entangled?
- What is the reduced density matrix for particle 1?
- Suppose an ensemble of systems with wave function $|\psi\rangle$. Suppose the spin of particle 2 is measured in the basis $(|0\rangle,\ |1\rangle)$ for each system. What is the resulting reduced density matrix for particle 1?
- Suppose an ensemble of systems with wave function $|\psi\rangle$. Suppose a unitary operation is applied to particle 2 for each system. What is the resulting reduced density matrix for particle 1?

4 Perturb (3pts)

Determine to first order in the parameter ϵ the corrections to the eigenvalues and to the eigenvectors of the following Hamiltonian:

$$\begin{pmatrix} a & \epsilon & 0 \\ \epsilon & b & 0 \\ 0 & 0 & a-b \end{pmatrix}, \tag{3}$$

with $a, b \in \mathbb{R}\backslash\{0\}$, $a \neq b$. Do that without solving the complete (unperturbed) eigenequation.

5 Think twice (6pts)

Answer with yes (true, i agree) or no (false, that is wrong):

a) entanglement is responsible for the interference pattern in a double-slit experiment.

b) the Aharonov-Bohm double-slit experiment shifts the interference pattern for light.

c) the commutator of a matrix A and its exponential $\exp A$ is not zero.

d) The Feynman path-integral depends on the classical Lagrangian.

e) The Rabi frequency decreases with the magnitude of the electromagnetic field.

f) The (first) Stern-Gerlach experiment dates from the 1930's.