Solutions of the test Adv QM, 4 November 2020

1 Approximate (3pts)

Consider the Hamiltonian (for $x \in \mathbb{R}$)

$$H_b = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right) + bx^4$$

Approximate its ground-state energy E_b in a reasonable way for $b > 0$. You could use a variational approach or anything else you judge appropriate.

The ground state of H_0 is $\psi_0(x) \propto \exp(-x^2/2)$. Let us try therefore the trial function $\psi_\sigma(x) \propto \exp(-x^2/2\sigma^2)$, with σ as variational parameter. We find that the new ground state energy must satisfy

$$E_b \leq \frac{(\sigma^{-4}+1)\sigma^2}{4} + b\frac{3\sigma^4}{4}$$

for all $\sigma > 0$. For large b you can try $\sigma = (1/6b)^{1/6}$ which minimizes $\frac{\sigma^{-2}}{4} + b\frac{3\sigma^4}{4}$. For small b we can use perturbation theory.

$$E_b - E_0 = \langle \psi_0, bx^4\,\psi_0\rangle = \frac{3b}{4}$$

in linear order.

2 Qubit (4pts)

a) Consider a single qubit. Show that any unitary operator U on $\mathbb{C}^2$ can be written as a rotation in spin space, up to a constant phase factor. That is, there exist $\alpha, \theta \in \mathbb{R}$ and a unit vector $\mathbf{n}$ such that:

$$U = \exp(\mathrm{i}\alpha)\exp(\mathrm{i}\theta\mathbf{n}\cdot\boldsymbol{\sigma}) \tag{1}$$

b) Calculate $\mathrm{Tr}[\,U\,\sigma_x]$.

a) The Pauli matrices are Hermitian and unitary; together with the unit matrix they span the set of 2×2 matrices with complex entries. Each unitary matrix $U = e^{iA}$ for some Hermitian matrix. Each Hermitian or unitary 2×2 matrix A can be written as a linear combination of the unit matrix and the three Pauli matrices.

b) Note that by expanding the exponential,

$$\exp(\mathrm{i}\theta\mathbf{n}\cdot\boldsymbol{\sigma}) = \cos\theta + \mathrm{i}\,\sin\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}$$

So we only need $\mathrm{Tr}[\sigma_\alpha\sigma_\beta] = 0$ when $\alpha \neq \beta$, $\sigma_\alpha^2 = I$ (identity).

3 Density matrix (4pts)

Consider the state

$$|\psi\rangle = a|01\rangle + b|10\rangle \tag{2}$$

with $|a|^2 + |b|^2 = 1$.

- For which values of a and b is $|\psi\rangle$ entangled?
  If $|\psi\rangle$ is separable, then there are $c, d, e, f \in \mathbb{C}$ such that $|\psi\rangle = (c|0\rangle + d|1\rangle)(e|0\rangle + f|1\rangle)$. Thence, $ce = 0, df = 0, cf = 0, de = b$. If $a \neq 0$ then $e = 0$ and hence $b = 0$. Similarly, if $b \neq 0$, then $a = 0$. Conclusion: $|\psi\rangle$ is entangled if and only if $a \neq 0, b \neq 0$.

- What is the reduced density matrix for particle 1?

$$\rho_1 = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1|$$

- Suppose an ensemble of systems with wave function $|\psi\rangle$. Suppose the spin of particle 2 is measured in the basis $(|0\rangle,\ |1\rangle)$ for each system. What is the resulting reduced density matrix for particle 1?
  Statistical mixture has also same density matrix ρ_1.

- Suppose an ensemble of systems with wave function $|\psi\rangle$. Suppose a unitary operation is applied to particle 2 for each system. What is the resulting reduced density matrix for particle 1?
  It means that $\rho \longrightarrow \rho' = (1 \otimes U^*)\rho(1 \otimes U)$. Taking partial trace $\mathrm{Tr}_2\rho' = \mathrm{Tr}_2(1 \otimes U^*)\rho(1 \otimes U) = \mathrm{Tr}_2\rho = \rho_1$.

4 Perturb (3pts)

Determine to first order in the parameter ϵ the corrections to the eigenvalues and to the eigenvectors of the following Hamiltonian:

$$H_0 + \epsilon H_1 = \begin{pmatrix} a & \epsilon & 0 \\ \epsilon & b & 0 \\ 0 & 0 & a-b \end{pmatrix} \tag{3}$$

with $a, b \in \mathbb{R}\backslash\{0\}$, $a \neq b$. Do that without solving the complete (unperturbed) eigenequation. For $\epsilon = 0$ the eigenstates are $(100)^T$ with eigenvalue a, $(010)^T$ with eigenvalue b and $(001)^T$ with eigenvalue $a - b$. There is a non-degenerate spectrum, and we have

$$E_n^{(1)} = \langle n|H_1|n\rangle = 0$$

for H_1 the perturbation. Also,

$$|\psi_n^{(1)}\rangle = \sum_{m\neq n} \frac{|m\rangle\langle m|H_1|n\rangle}{E_n^0 - E_m^0}$$

so that $|\psi_1^{(1)}\rangle = |2\rangle/(a-b),\ |\psi_2^{(1)}\rangle = |1\rangle/(b-a),\ |\psi_3^{(1)}\rangle = 0.$

5 Think twice (6pts)

Answer with yes (true, i agree) or no (false, that is wrong):

a) entanglement is responsible for the interference pattern in a double-slit experiment. NO
b) the Aharonov-Bohm double-slit experiment shifts the interference pattern for light. NO
c) the commutator of a matrix A and its exponential $\exp A$ is not zero. NO
d) The Feynman path-integral depends on the classical Lagrangian. YES
e) The Rabi frequency decreases with the magnitude of the electromagnetic field. NO
f) The (first) Stern-Gerlach experiment dates from the 1930's. NO